Exercise Sheet: Lecture 1
Projective Spaces over Finite Fields

Michel Lavrauw

GAP Days Spring 2025, VUB

Exercises

1. Construct the projective plane $\mathrm{PG}(2,3)$ in GAP using the FinInG package.
   (a) Pick a point in $\mathrm{PG}(2,3)$ and list all lines that pass through it.
   (b) Verify that each line in $\mathrm{PG}(2,3)$ contains exactly 4 points and that each point lies on exactly 4 lines.
   (c) An *arc* is a set of points, no three of which are collinear. Construct an arc of size 4.
   (d) Verify whether you can construct an arc of size 5.

2. Construct the projective plane $\mathrm{PG}(2,4)$.
   (a) Write a function (IsArc) which determines whether a given set of points is an arc.
   (b) Determine the largest size of an arc in $\mathrm{PG}(2,4)$.
   (c) A arc is called *complete* if it is not contained in a larger arc. Determine the possible sizes of complete arcs in $\mathrm{PG}(2,4)$.

3. A conic in $\mathrm{PG}(2,q)$ is the zero locus $\mathcal{Z}(f)$ of a quadratic form in $\mathrm{GF}(q)[X_0,X_1,X_2]$.
   (a) Use the function *QuadraticVariety* to write function which constructs a random conic in $\mathrm{PG}(2,q)$.
   (b) Determine the spectrum of number of points of conics in $\mathrm{PG}(2,16)$.
   (c) Find a conic $\mathcal{C}$ in $\mathrm{PG}(2,16)$, whose points form an arc $\mathcal{A}$ (use the IsArc function you wrote).
   (d) For each point of $\mathcal{C}$ determine the *tangent line* to $\mathcal{C}$ through that point.
   (e) Show that the tangents of $\mathcal{C}$ are collinear.
   (f) Determine a complete arc containing the arc $\mathcal{A}$.

4. Construct the projective space $\mathrm{PG}(3,7)$.

(a) Construct a set of three pairwise disjoint lines A, B, and C.

(b) Given a set of lines S, a line meeting all lines of S is called a *transversal of S*. Construct the set of all transversals of the set of lines A, B, C.

(c) Find a line D in $\mathrm{PG}(3,7)$, such that the set of lines $\{A, B, C, D\}$ has exactly two transversals.

5. Construct the projective space $\mathrm{PG}(3,5)$. A *cap* in $\mathrm{PG}(3,q)$ is a set of points no three of which on a line. A cap $\mathcal{A}$ is *complete* if $\mathcal{A}$ is not contained in a larger cap.

   (a) Write a function (IsCap) which checks whether a set of points is a cap.

   (b) Determine the largest size of a cap in $\mathrm{PG}(3,5)$.

   (c) Determine the smallest size of a complete cap in $\mathrm{PG}(3,5)$.

6. A *blocking set* in $\mathrm{PG}(2,q)$ is a set of points, which contains at least one point of each line of $\mathrm{PG}(2,q)$. A point P in a blocking set $\mathcal{B}$ is *essential* if the set $\mathcal{B} \setminus \{P\}$ is not a blocking set. A blocking set is called *minimal* if each of its points is *essential*. A blocking set $\mathcal{B}$ is called *trivial* it $\mathcal{B}$ contains a line.

   (a) Write a function (IsBlockingSet) which checks whether a set of points is a blocking set.

   (b) Determine the size of a non-trivial minimal blocking set in $\mathrm{PG}(2,9)$.

   (c) Determine the largest size of a non-trivial minimal blocking set in $\mathrm{PG}(2,9)$.

7. In this exercise, we will study linear sets in $\mathrm{PG}(2,8)$. Use the method NaturalEmbeddingByFieldReduction provided in FinInG. A $\mathrm{GF}(q)$-linear set of rank k is called *scattered* if its size is $(q^k-1)/(q-1)$.

   (a) Write a function which constructs a random $\mathrm{GF}(2)$-linear set in $\mathrm{PG}(2,8)$ of given rank k.

   (b) Construct a non-trivial $\mathrm{GF}(2)$-linear blocking set in $\mathrm{PG}(2,8)$.

   (c) Determine the maximum rank of a scattered $\mathrm{GF}(2)$-linear set in $\mathrm{PG}(2,8)$.

   (d) Determine the maximum size of a $\mathrm{GF}(2)$-linear set in $\mathrm{PG}(2,8)$.

   (e) Determine the spectrum of the sizes of $\mathrm{GF}(2)$-linear sets in $\mathrm{PG}(2,8)$.

8. In this exercise, we will use the method HermitianVariety

   (a) Construct a Hermitian curve in $\mathrm{PG}(2,9)$.

   (b) Determine the intersection numbers of H with lines of $\mathrm{PG}(2,9)$.

9. In this exercise, we will use the method SegreVariety.

(a) Construct the Segre threefold $\mathrm{PG}(1,3) \times \mathrm{PG}(2,3)$ in $\mathrm{PG}(5,3)$ (notation $S_{1,2}$).

(b) Determine the set-wise stabiliser G of the points of $S_{1,2}$ inside the projectivity group of $\mathrm{PG}(5,3)$.

(c) Determine the orbits $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$of G on the set of points of $\mathrm{PG}(5,3)$.

(d) Determine the orbits of G on lines of $\mathrm{PG}(5,3)$.

(e) Given a line L, the *point-orbit distribution* of L (notation $OD_0(L)$) is a list of integers $[a_1, a_2, \ldots, a_t]$ where a_i is the number of points of L which belong to $\mathcal{P}_i$. Write a function which computes the *point-orbit distribution* of a line in $\mathrm{PG}(5,3)$).

(f) Use the natural correspondence between $S_{1,2}$ and the projective space of 2×3-matrices over $\mathrm{GF}(q)$, to write a function which maps each point of $\mathrm{PG}(5,q)$ to a 2×3 matrix over $\mathrm{GF}(q)$.

10. A quadric $\mathrm{PG}(n,q)$ is the zero locus of a quadratic form $f \in \mathrm{GF}(q)[X_0, \ldots, X_n]$.

(a) Use the function *QuadraticVariety* to write function which constructs a random quadric in $\mathrm{PG}(n,q)$.

(b) Determine the spectrum of number of points of quadrics in $\mathrm{PG}(3,7)$.

(c) Link this spectrum to the types of the quadric forms (use *TypeOfForm* and *QuadraticForm*).