Lecture 1. Projective Spaces over Finite Fields (FinInG package)

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Outline

If you want to find out more about the University of Primorska:

https://conferences.famnit.upr.si/event/33/overview

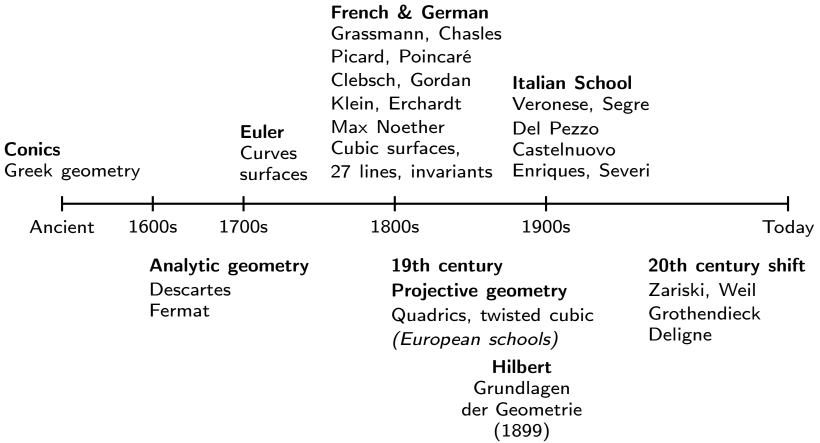
Introduction

Classical (Algebraic) Geometry: A Starting Point

- **Conics in the Plane**: The first nontrivial algebraic curves.
- **Quadrics in Projective Space**: Including ellipsoids, hyperboloids, cones.
- **Twisted Cubic**: The simplest non-degenerate rational space curve.
- **Normal Rational Curves (NRC)**: Embedding $\mathbb{P}^1$ into $\mathbb{P}^d$.
- **Veronese Varieties**: Embedding $\mathbb{P}^n$ into $\mathbb{P}^N$.
- **Segre Varieties**: Product embeddings $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$.
- **Plane Cubics**: Groups from curves.
- **Cubic Surfaces**: 27 lines, a historical gem of 19th century geometry.

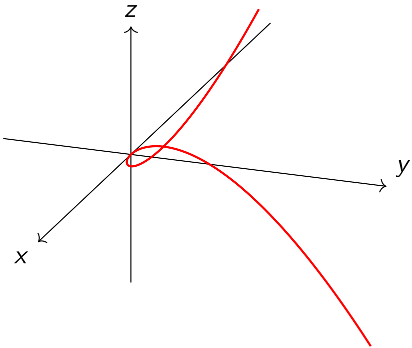
These classical objects are still being studied today in many different settings.

Timeline: From Classical Geometry to Modern Algebraic Geometry



Twisted Cubic in $\mathbb{P}^3(\mathbb{R})$

$$\mathbb{P}^1 \to \mathbb{P}^3 \;:\; [s:t] \mapsto [s^3 : s^2 t : st^2 : t^3]$$



(affine part)

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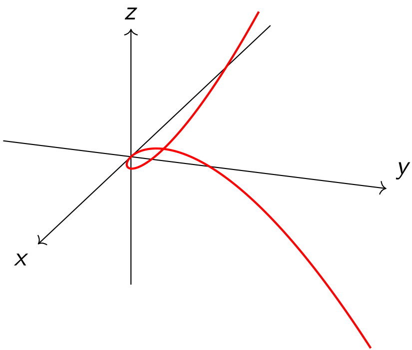

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IMPORTANT! No 4 points in a plane → an arc in $\mathbb{P}^3$

Timeline: Foundations and Geometry over Finite Fields

A. Young
Gaussian coefficients
Counting subspaces
of a $\mathrm{GF}(q)^n$

Veblen & Young
Projective Geometry
over $\mathrm{GF}(q)$

Segre
Geometria di Galois,
arcs and caps,
algebraic curves,
polarities

Tits
Buildings,
Polar spaces,
Groups & Geometries,
Incidence structures

1901 — 1910 — 1930s — 1950s — 1960s — 1970s — 1990s — Today

Steinitz, **Albert**, **André**, **Marshall Hall Jr.**,
Field-based foundations, Coordinatisation,
Planary Ternary Ring, Translation planes,
Quasifields, Nearfields, Semifields, Spreads

André/Bruck-Bose
The construction
of translation
planes from
projective spaces

Modern era
Galois Geometry,
Incidence Geometry,
Design Theory,
+ Applications

Hilbert
(1899)

Moulton
(1902)

Hessenberg
(1905)

Early
foundational
figures

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  A conic or cubic over $\mathbb{F}_q$ has only a finite number of points.
- × **Lines do not always intersect curves**
  Over finite fields, we do not know how curves intersect
  (Bézout's theorem holds over algebraically closed fields)

WARNING! Finite Fields detected. Classical intuition disabled.

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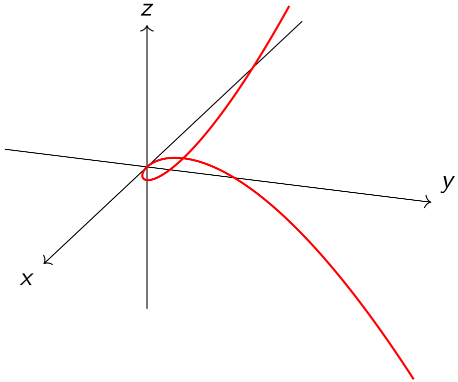
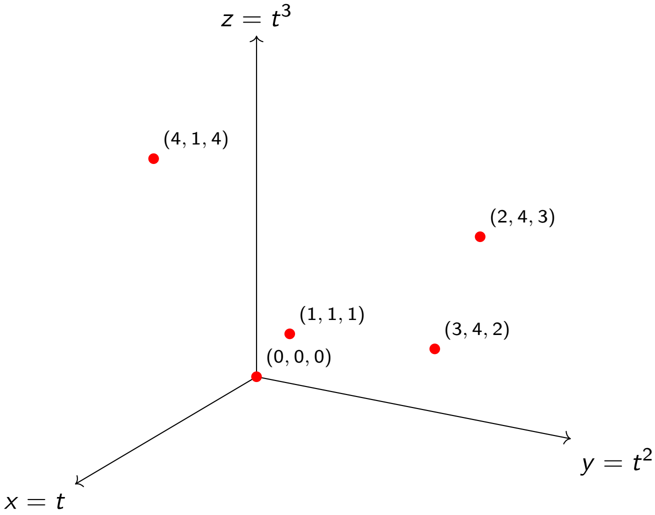
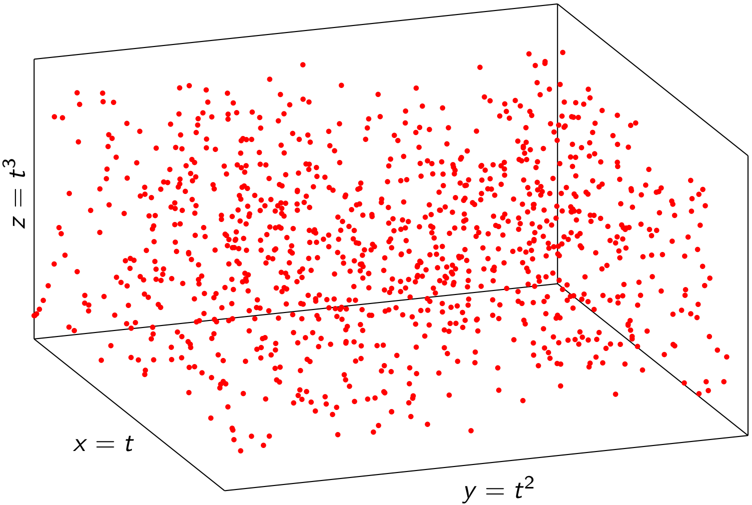


Twisted Cubic in $\mathbb{P}^3(\mathrm{GF}(5))$



Twisted Cubic in $\mathbb{P}^3(\mathrm{GF}(997))$



Projective spaces in FinInG

Using FinInG to Construct $\mathrm{PG}(n, q)$

- Load the package: `LoadPackage("fining");`
- Create the projective space:
  `pg := PG(2, 3);`
- Create subspaces: `VectorSpaceToElement`
- Random point: `Random(Points(pg));`
- Random line: `Random(Lines(pg));`
- The span of subspaces: `l := Span(p1, p2);`
- Incidence check: `p in l;`
- All k-dimensional subspaces:
  `ElementsOfIncidenceStructure(pg, k+1);`

A first example

```
gap> pg:=PG(2,2);
ProjectiveSpace(2, 2)
gap> Points(pg);
<points of ProjectiveSpace(2, 2)>
gap> List(Points(pg),x->Coordinates(x));
[ [ Z(2)^0, 0*Z(2), 0*Z(2) ], [ Z(2)^0, 0*Z(2), Z(2)^0 ],
  [ Z(2)^0, Z(2)^0, 0*Z(2) ], [ Z(2)^0, Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), Z(2)^0, 0*Z(2) ], [ 0*Z(2), Z(2)^0, Z(2)^0 ],
  [ 0*Z(2), 0*Z(2), Z(2)^0 ] ]
gap> Display(last);
 1 . .
 1 . 1
 1 1 .
 1 1 1
 . 1 .
 . 1 1
 . . 1
```

```
gap> Lines(pg);
<lines of ProjectiveSpace(2, 2)>
gap> Size(last);
7
gap> line:=Random(Lines(pg));
<a line in ProjectiveSpace(2, 2)>
gap> ProjectiveDimension(line);
1
gap>
gap> UnderlyingObject(line);
<immutable cmat 2x3 over GF(2,1)>
gap> Display(last);
[[1..]
 [.11]
]
```

Affine spaces

```
AG(8, 5)
gap> Random(Planes(ag));
<a plane in AG(8, 5)>
gap> Display(last);
Affine plane:
Coset representative:
NewVector(IsCVecRep,GF(5,1),[0*Z(5),0*Z(5),Z(5)^0,
0*Z(5),Z(5)^2,Z(5)^3,Z(5)^2,Z(5),])
Coset (direction): NewMatrix(IsCMatRep,GF(5,1),8,[
[ Z(5)^0, 0*Z(5), Z(5)^0, 0*Z(5), Z(5), Z(5)^2, Z(5), Z(5)^0 ],
[ 0*Z(5), Z(5)^0, 0*Z(5), 0*Z(5), Z(5)^3, Z(5), Z(5)^3, 0*Z(5) ],])
gap>
```

More Examples

See file `1_gapcode_ML.g`.

Summary

- FinInG allows easy construction and manipulation of objects in projective spaces over finite fields.
- We can explore geometric properties of points, lines, subspaces, incidence, groups actions, ...

Discussion Questions

- How are projective spaces used in applications (coding theory)?
- What other geometries can be built in FinInG?