Exercise Sheet: Lecture 3
Incidence Structures in FinInG

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Exercises

1. Consider a conic $\mathcal{C}$ in the projective plane $\mathrm{PG}(2,5)$.
   (a) Embed the projective plane as a hyperplane H in a 3-dimensional projective space $\Sigma \cong \mathrm{PG}(3,5)$.
   (b) Define the incidence structure S with points $\mathcal{P}$ and lines $\mathcal{L}$ consisting of the following types of "points" and "lines":

   "points": (a) the points of Σ which are not in H, (b) the planes of Σ which meet H in a tangent line to $\mathcal{C}$, (c) a special point (∞);

   "lines": (i) lines of Σ meeting H in a point of $\mathcal{C}$, (ii) the points of $\mathcal{C}$.

   The incidence relation relation is defined as the natural incidence relation in the projective space Σ, and the point (∞) is incident with all the lines of type (b), and not incident with any points of type (a).
   (c) Show that the incidence structure S is a generalised quadrangle of order $(s,t) = (5,5)$.
   (d) Show that S is isomorphic to the generalised quadrangle $Q(4,5)$.
2. Consider a 3-dimensional space $\Omega \cong \mathrm{PG}(3,7)$.
   (a) Define the Field Reduction map ϕ from $\mathrm{PG}(1,7^2)$ to Ω (NaturalEmbeddingByFieldReduction).
   (b) Show that the image of the set of points of $\mathrm{PG}(1,7^2)$ under ϕ defines a set S of pairwise disjoint lines in $\mathrm{PG}(3,7)$ which form a partition of the set of points of $\mathrm{PG}(3,7)$.
   (c) Embed the Ω as a hyperplane H in a 4-dimensional projective space $\Sigma \cong \mathrm{PG}(4,7)$.

(d) Define the following incidence geometry Δ with points $\mathcal{P}$ and lines $\mathcal{L}$ consisting of the following "points" and "lines":

"points": the points of Σ which are not in H,

"lines": planes of Σ which meet H in an element of S.

The incidence relation is the natural incidence in Σ. The type function sends points to 1 and lines to 2.

(e) Which classical geometry is Δ isomorphic to?