Linear complete symmetric rank-distance codes

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Symmetric Rank-distance codes

- An $\mathbb{F}_q$*-linear symmetric rank-distance code* is a subspace $C \leq S_n(\mathbb{F}_q)$ equipped with the rank-distance metric:

$$d(A, B) = \mathrm{rank}(A - B), \quad \text{for } A, B \in C.$$

- The *minimum distance* of C is:

$$d = d(C) = \min\{d(X, Y) : X, Y \in C, \ X \neq Y\}.$$

$\mathbb{F}_q$-linear CSRD codes:

- $\mathbb{F}_q$-linear SRD codes.
- An $\mathbb{F}_q$-linear code of min. distance d is *complete* if not contained in a larger $\mathbb{F}_q$-linear code with the same d.

Bounds on additive SRD codes

- [Schmidt, 2015]:

$$\dim(C) \leq \begin{cases} \frac{n(n-d+2)}{2} & \text{if } n-d \text{ is even,} \\ \frac{(n+1)(n-d+1)}{2} & \text{if } n-d \text{ is odd.} \end{cases} \quad (1)$$

- Constructions of additive SRD codes can be found in [Schmidt, 2015, 2020].
- Bound achieved $\Rightarrow$ MSRD code (*maximum symmetric rank distance*).
- Sphere-packing bound is achieved: SRD perfect codes $\iff$ n odd and $d = 3$ [Mushrraf and Zullo, 2025].

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To characterize $\mathbb{F}_q$-linear CSRD codes in $M_{n\times n}(\mathbb{F}_q)$ that are **not** MSRD codes.

To classify and characterize $\mathbb{F}_q$-linear CSRD codes in $M_{3\times 3}(\mathbb{F}_q)$.

GEOMETRIC INTERPRETATION

- An $\mathbb{F}_q$-linear SRD code in $M_{n\times n}(\mathbb{F}_q) \iff$ a subspace of the projective space $\mathrm{PG}(N-1,q)$; $N=\frac{n(n+1)}{2}$.
- The set of symmetric rank-one matrices in $M_{n\times n}(\mathbb{F}_q) \iff$ the set of points of the *Veronese variety* $\mathcal{V}_n(\mathbb{F}_q)$ in $\mathrm{PG}(N-1,q)$.
- $\mathcal{V}_n(\mathbb{F}_q)$ is the image of the *Veronese embedding*:

$$\nu_n : \mathrm{PG}(n,q) \to \mathrm{PG}(N-1,q)$$
$$((x_0,\ldots,x_{n-1}) \longmapsto (x_0^n, x_0^{n-1}x_1, \cdots, x_0^{n-1}x_{n-1}, \cdots, x_0^{n-2}x_1x_2, \cdots, x_{n-1}^n).$$

- $\mathcal{V}_n^{(d-1)}(\mathbb{F}_q) \iff$ the set of symmetric matrices in $M_{n\times n}(\mathbb{F}_q)$ of rank at most d.
- A k-dimensional CSRD code in $M_{n\times n}(\mathbb{F}_q)$ of minimum distance $d \iff$ a $(k-1)$-dimensional subspace of $\mathrm{PG}(N-1,q)$ that is disjoint from $\mathcal{V}_n^{(d-1)}(\mathbb{F}_q)$ (minimum rank d) and maximal for this property.

EQUIVALENCE

- C is *linear equivalent* to $C' \iff \exists\,(A,B) \in \mathrm{GL}(n,q) \times \mathrm{GL}(n,q)$ such that $C' = ACB$.
- C is linear equivalent to $C' \iff W$ is G-equivalent to W' in $\mathrm{PG}(n^2-1,q)$; $G \leqslant \mathrm{PGL}(n^2,q)$ stabilising $\mathcal{S}_{n,n}(\mathbb{F}_q)$ the image of the *Segre embedding*:

$$\sigma_{n,n} : \mathrm{PG}(n-1,q) \times \mathrm{PG}(n-1,q) \longrightarrow \mathrm{PG}(n^2-1,q)$$

$$((x_0,\ldots,x_{n-1}),(y_0,\ldots,y_{n-1})) \longmapsto (x_0y_0, x_0y_1,\ldots,x_iy_j,\ldots,x_{n-1}y_{n-1})\,.$$

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Symmetric rank-distance codes:

- C is *symmetric equivalent* to $C' \iff \exists\, X \in \mathrm{GL}(n,q) \times \mathrm{GL}(n,q)$ such that $C' = XCX^T$.
- C is *symmetric equivalent* to $C' \iff W$ is K-equivalent to W'; $K \leqslant \mathrm{PGL}(\frac{n(n+1)}{2},q)$ stabilising $\mathcal{V}_n(\mathbb{F}_q)$.

Equivalence classes of CSRD codes: 3×3 case

$d = 2$:

- **Theorem 1:**
  There are 3 equivalence classes of $\mathbb{F}_q$-linear CSRD codes in $M_{3\times 3}(\mathbb{F}_q)$ of minimum distance 2 for q odd. Moreover, each $\mathbb{F}_q$-linear CSRD code in $M_{3\times 3}(\mathbb{F}_q)$, q odd, of minimum distance 2 is an MSRD code.
- **Theorem 2:**
  There are 6 equivalence classes of $\mathbb{F}_q$-linear CSRD codes in $M_{3\times 3}(\mathbb{F}_q)$ of minimum distance 2 for $q \geq 4$ even, 3 of which are MSRD codes.

$d = 3$:

- MSRD codes in $M_{3\times 3}(\mathbb{F}_q)$ with $d = 3$ have dimension $3 \to$ MRD codes: Equivalence classes of MRD codes in $M_{3\times 3}(\mathbb{F}_q)$ with $d = 3 \iff$ the isotopism classes of 3-dimensional semifields.
- A 3-dimensional semifield is either a field or isotopic to a twisted field [Menichetti, 1977].
- **Lemma:** An $\mathbb{F}_q$-linear CSRD code in $M_{3\times 3}(\mathbb{F}_q)$ with $d = 3$ is an MSRD code.

Approach

A k-dimensional CSRD code in $M_{3\times 3}(\mathbb{F}_q)$ of minimum distance $d \iff$ a $(k-1)$-dimensional subspace of $\mathrm{PG}(5,q)$ that is disjoint from $\mathcal{V}^{(d-1)}(\mathbb{F}_q)$ (minimum rank d) and maximal for this property.

Representation:

The solid in $\mathrm{PG}(5,q)$ spanned by the 1st four points of the standard frame is

$$\begin{bmatrix} x & y & z \\ y & t & . \\ z & . & . \end{bmatrix} := \{ \begin{bmatrix} x & y & z \\ y & t & 0 \\ z & 0 & 0 \end{bmatrix} : (x,y,z,t) \in \mathbb{F}_q^4;\ (x,y,z,t) \neq (0,0,0,0)\}.$$

$d = 2$:

To determine solids (MSRD), planes, and lines of minimum rank 2 that are disjoint from $\mathcal{V}(\mathbb{F}_q)$ and maximal for this property.

$d = 3$:

To determine planes (MSRD) and lines of minimum rank 3 that are disjoint from $\mathcal{V}^{(2)}(\mathbb{F}_q)$ and maximal for this property.

ORBITS INVARIANTS:

- The **rank distribution of** W is

$$[r_1, r_2, r_3],$$

where

$$r_i = \# \text{ of rank } i \text{ points in } W.$$

$C \iff W$:

The **rank distribution of** C is $(1, r_1(q-1), r_2(q-1), r_3(q-1))$.

- The r**-space orbit-distribution of** W is

$$OD_r(W) = [u_1, u_2, \ldots, u_m],$$

where

u_i = # of r-spaces incident with W which belong to the orbit U_i (U_1, U_2,...,U_m are the distinct K-orbits of r-spaces in $\mathrm{PG}(5, q)$).

Points and hyperplanes of $\mathrm{PG}(5,q)$

K-orbits of points; q odd / even:

- $\mathcal{P}_1$:= Rank-one points.
- $\mathcal{P}_{2,e}$:= Exterior rank-two points / $\mathcal{P}_{2,n}$:= Rank-two points in the *nucleus plane*.
- $\mathcal{P}_{2,i}$:= Interior rank-two points / $\mathcal{P}_{2,s}$:= Rank-two points outside the nucleus plane.
- $\mathcal{P}_3$:= Rank-three points.

K-orbits of Hyperplanes:

- $\mathcal{H}_1$:={ Hyperplanes $\iff$ repeated lines in $\mathrm{PG}(2,q)$}.
- $\mathcal{H}_{2,r}$:= { Hyperplanes $\iff$ pairs of real lines in $\mathrm{PG}(2,q)$}.
- $\mathcal{H}_{2,i}$:= { Hyperplanes $\iff$ pairs of conjugate imaginary lines in $\mathrm{PG}(2,q^2)$}.
- $\mathcal{H}_3$:= { Hyperplanes $\iff$ non-singular conics in $\mathrm{PG}(2,q)$}.

Solids of minimum rank 2

K-orbits of solids in $\mathrm{PG}(5,q)$

- For q odd: there are 15 K-orbits of solids corresponding to the K-orbits of lines in $\mathrm{PG}(5,q)$ ([Lavrauw and Popiel, 2020]).
  (MSRD $\iff$ solids in $\Omega_{8,2} \cup \Omega_{14,2} \cup \Omega_{15,2}$).
- For q even: there are 15 K-orbits of solids as determined in [A., Lavrauw and Popiel, 2022].

S^K	Representatives	$OD_0(S)$	$OD_4(S)$
Ω_7	$\begin{bmatrix} x & y & z \\ y & x+\gamma y & t \\ z & t & y \end{bmatrix}$	$[0, q+1, q^2+q, q^3-q]$	$[1, 0, 1, q-1]$
Ω_{13}	$\begin{bmatrix} x & y & z \\ y & \gamma x + y & t \\ z & t & \gamma x + z \end{bmatrix}$	$[0, 1, q^2+3q, q^3-2q]$	$[0, 1, 2, q-2]$
Ω_{14}	$\begin{bmatrix} x & y & \gamma x + y + \gamma t \\ y & \gamma x + y & z \\ \gamma x + y + \gamma t & z & t \end{bmatrix}$	$[0, 1, q^2+q, q^3]$	$[0, 0, 1, q]$

Table: Geometric interpretation of the 3 equivalence classes of $\mathbb{F}_q$-linear MSRD codes in $M_{3\times 3}(\mathbb{F}_q)$ for q even, where $\mathrm{Tr}(\gamma^{-1}) = 1$ in Ω_7 and $\mathrm{Tr}(\gamma) = 1$ in Ω_{13} and Ω_{14}.

PLANES OF MINIMUM RANK 2

Theorem:

- For q odd, a plane in $\mathrm{PG}(5, q)$ of minimum rank 2 is contained in a solid in $\Omega_{8,2} \cup \Omega_{14,2} \cup \Omega_{15,2}$.
- For $q \geq 4$ even, a plane in $\mathrm{PG}(5, q)$ of minimum rank 2 with at least one rank-2 point outside the nucleus plane $\pi_{\mathcal{N}}$ is contained in a solid in $\Omega_7 \cup \Omega_{13} \cup \Omega_{14}$.

Proof.

Let π be such a plane and $P \in \pi \cap \mathcal{V}^{(2)}(\mathbb{F}_q)$. The number of solids through π in $\mathrm{PG}(5, q)$ is equal to the number of points on $\mathcal{V}(\mathbb{F}_q)$, it follows that there must be at least one solid through π which does not meet $\mathcal{V}(\mathbb{F}_q)$, unless all solids of the form $\langle \pi, Q \rangle$, with $Q \in \mathcal{V}(\mathbb{F}_q)$ are distinct. To see that this is not the case, consider the conic $\mathcal{C}(P)$ and a secant line $\langle Q, Q' \rangle$ of $\mathcal{C}(P)$ through P. Then $\langle \pi, Q \rangle = \langle \pi, Q' \rangle$.

For q even, the existence of a secant line through P is guaranteed, as π contains at least one rank-2 point outside $\pi_{\mathcal{N}}$.

□

$\pi \cap \mathcal{V}^{(2)}(\mathbb{F}_q) \subseteq \pi_{\mathcal{N}}$:

- $\pi = \pi_{\mathcal{N}}$, $\pi \cap \pi_{\mathcal{N}} = \ell \in o_{12,1}$ or $\pi \cap \pi_{\mathcal{N}} = P$ (→ 3 distinct K-orbits: $\Sigma_{\mathcal{N}}$, Σ and Σ').

Theorem:

For a plane π in $\mathrm{PG}(5, q)$ with $q \geq 4$ even, we have $r_{2,n}(\pi) = h_1(\pi)$; that is, the number of hyperplanes in $\mathcal{H}_1$ containing π equals $|\pi \cap \pi_{\mathcal{N}}|$.

- **Lemma:** For $\pi \in \Sigma$, $OD_4(\pi) = [q + 1, 0, 0, q^2]$.
- **Lemma:** For $\pi \in \Sigma'$, $OD_4(\pi) = [1, 0, 0, q^2 + q]$.

Theorem:

For $q \geq 4$ even, a plane in $\mathrm{PG}(5, q)$ disjoint from $\mathcal{V}(\mathbb{F}_q)$ and intersecting $\mathcal{V}^{(2)}(\mathbb{F}_q)$ in a nonempty subspace of $\pi_{\mathcal{N}}$ is maximal with respect to that property, i.e., it is not contained in any solid of minimum rank 2.

K-orbits of Lines in $\mathrm{PG}(5, q)$

Orbits	$[r_1, r_{2,e}, r_{2,i}, r_3]$
o_5	$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$
o_6	$[1, q, 0, 0]$
$o_{8,1}$	$[1, 1, 0, q-1]$
$o_{8,2}$	$[1, 0, 1, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$
$o_{12,1}$	$[0, q+1, 0, 0]$
$o_{13,1}$	$[0, 2, 0, q-1]$
$o_{13,2}$	$[0, 1, 1, q-1]$
$o_{14,1}$	$[0, 3, 0, q-2]$
$o_{14,2}$	$[0, 1, 2, q-2]$
$o_{15,1}$	$[0, 1, 0, q]$
$o_{15,2}$	$[0, 0, 1, q]$
$o_{16,1}$	$[0, 1, 0, q]$
o_{17}	$[0, 0, 0, q+1]$

Orbits	$[r_1, r_{2,n}, r_{2,s}, r_3]$
o_5	$[2, 0, q-1, 0]$
o_6	$[1, 1, q-1, 0]$
$o_{8,1}$	$[1, 0, 1, q-1]$
$o_{8,3}$	$[1, 1, 0, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, 0, q+1, 0]$
$o_{12,1}$	$[0, q+1, 0, 0]$
$o_{12,3}$	$[0, 1, q, 0]$
$o_{13,1}$	$[0, 1, 1, q-1]$
$o_{13,3}$	$[0, 0, 2, q-1]$
$o_{14,1}$	$[0, 0, 3, q-2]$
$o_{15,1}$	$[0, 0, 1, q]$
$o_{16,1}$	$[0, 1, 0, q]$
$o_{16,3}$	$[0, 0, 1, q]$
o_{17}	$[0, 0, 0, q+1]$

Table: K-orbits of lines in $\mathrm{PG}(5, q)$ for q odd (left) and q even (right) [Lavrauw, Popiel, 2020]

Lines of minimum rank 2:

Lemma:

A line in $\mathrm{PG}(5,q)$ of minimum rank 2 extends to a solid in $\Omega_{8,2} \cup \Omega_{14,2} \cup \Omega_{15,2}$ for q odd and $\Omega_7 \cup \Omega_{13} \cup \Omega_{14}$ for q even.

Sketch of the proof:

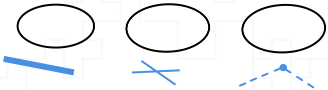
1. ℓ is contained in at least q^3 planes of minimum rank 2.
2. q odd $\longrightarrow \ell \in \Omega_{8,2} \cup \Omega_{14,2} \cup \Omega_{15,2}$.
3. q even: If $\ell \cap \mathcal{V}^2(\mathbb{F}_q) \not\subset \pi_{\mathcal{N}} \longrightarrow \ell \in \Omega_7 \cup \Omega_{13} \cup \Omega_{14}$. Else, $\ell \in o_{12,1} \cup o_{16,1}$. A solid $S \in \Omega_7$ contains a line in $o_{12,1}$ and $o_{16,1}$.

Implications on nets of conics in $\mathrm{PG}(2, q)$

- K-orbits of subspaces in $\mathrm{PG}(5, q) \iff \mathrm{PGL}(3, q)$-orbits of linear systems of conics in $\mathrm{PG}(2, q)$.
- **Open problem:** Classifying nets of conics in $\mathrm{PG}(2, q)$ with empty bases $\iff$ Classifying planes in $\mathrm{PG}(5, q)$ disjoint from $\mathcal{V}(\mathbb{F}_q)$.
- There is a unique $\mathrm{PGL}(3, q)$-orbit of nets of conics in $\mathrm{PG}(2, q)$ with conic distribution $[1, 0, 0, q^2 + q]$ for $q > 2$ even, represented by

$$\mathcal{N}_{18} = \alpha(cX_0X_2 + X_1^2) + \beta(X_0^2 + X_0X_2 + X_1X_2) + \gamma X_2^2;$$

  c is a non-admissible element of $\mathbb{F}_q$ satisfying $\mathrm{Tr}(c^{-1}) = \mathrm{Tr}(1)$
- Non-trivial perfect $\mathbb{F}_q$-linear CSRD codes in $M_{3\times 3}(\mathbb{F}_q)$ correspond to nets of conics in $\mathrm{PG}(2, q)$ that exclude singular conics.
- A net of conics with an empty base and at least one pair of lines (real or imaginary) contains a pencil of conics with an empty base:



Final Remarks

- For q odd: All $\mathbb{F}_q$-linear SRD codes in $M_{3\times 3}(\mathbb{F}_q)$ are extendable to MSRD codes.
- For q even: The only $\mathbb{F}_q$-linear SRD codes in $M_{3\times 3}(\mathbb{F}_q)$ that cannot extend to an MSRD code are the 3-dimensional CSRD codes with rank distributions $(1, 0, q^3 - 1, 0)$, $(1, 0, q^2 - 1, q^3 - q^2)$ and $(1, 0, q - 1, q^3 - q)$.
- While every MSRD code is a CSRD code, obtaining the conditions under which the converse fails remains an open question, particularly for $n > 3$.
- $\mathbb{F}_q$-linear CSRD codes in $M_{n\times n}(\mathbb{F}_q) \iff$ Linear systems of quadrics in $\mathrm{PG}(n-1, q)$.

Thank you for your attention!

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