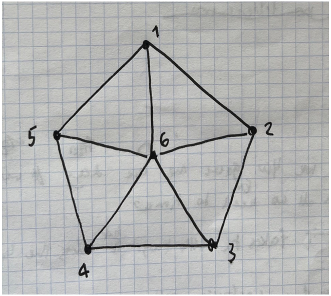
Codes, geometries and entangled quantum states associated with stabiliser codes

Simeon Ball



This is a regular hyperoval in PG(2, 4).

$$\mathrm{H} = \left(\begin{array}{cccccc||cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right).$$

Let H be a $(n-k) \times 2n$ matrix over $\mathbb{F}_q$ of rank $n-k$.

Let ℓ_i be the line of $\mathrm{PG}(n-k-1, q)$ spanned by the i-th and $(i+n)$-th column, for $i = 1, \ldots, n$.

The wheel graph gives a set of 6 lines in $PG(5, 2)$ which are obtained from the regular hyperoval of $\mathrm{PG}(2, 4)$ by field reduction.

Let

$$\Lambda = \left(\begin{array}{c||c} 0_n & \mathrm{I}_n \\ \hline\hline -\mathrm{I}_n & 0_n \end{array} \right).$$

Then

$$\mathrm{H}\Lambda\mathrm{H}^T = 0.$$

iff row space of H is totally isotropic wrt the symplectic form

$$(x, y) = x\Lambda y^T = \sum_{i=1}^{n} x_i y_{i+n} - y_i x_{i+n}.$$

Mapping

$$(a_i, b_i) \mapsto (-b_i, a_i)$$

preserves the symplectic product since

$$a_i b_i' - a_i' b_i = (-b_i)a_i' - (-b_i')a_i.$$

So we can change basis for the lines ℓ_i and reorder them so that

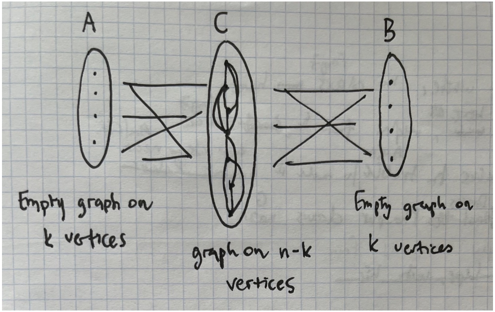
$$\mathrm{H} = \left(\begin{array}{c|c||c|c} \mathrm{I}_{n-k} & \mathrm{A} & \mathrm{C} + \mathrm{A}\mathrm{B}^T & \mathrm{B} \end{array} \right).$$

[Ball-Vilar 2025] [Adv Math.Comm.]

$$\mathrm{H}\Lambda\mathrm{H}^T = 0.$$

iff $\mathrm{C} = \mathrm{C}^T$.

So any symplectic self-orthogonal subspace of $PG(n-k-1,q)$ is equivalent to a graph on $n+k$ vertices.



$$\mathrm{H} = \left(\ \mathrm{I}_{n-k} \mid \mathrm{A} \parallel \mathrm{C} + \mathrm{A}\mathrm{B}^T \mid \mathrm{B}\ \right)$$

Let $\mathcal{X}$ be a quantum set of n lines in $\mathrm{PG}(n-1,p)$ and let π be a $(k-1)$-dimensional subspace skew to the lines of $\mathcal{X}$.

[Ball-Vilar 2025] The projection of $\mathcal{X}$ from π is a quantum set of n lines $\mathcal{X}_{\text{proj}}$ in $\mathrm{PG}(n-k-1,p)$.

If

$$\mathrm{H} = \left(\begin{array}{c|c||c|c} \mathrm{I}_{n-k} & 0 & \mathrm{A}_1 & \mathrm{A}_2 \\ \hline 0 & \mathrm{I}_k & \mathrm{A}_2^T & \mathrm{A}_4 \end{array}\right)$$

is the matrix associated to $\mathcal{X}$, then the matrix associated by $\mathcal{X}_{\text{proj}}$ is

$$\mathrm{H}_{\text{proj}} = \left(\begin{array}{c|c||c|c} \mathrm{I}_{n-k} & \mathrm{D}_1 & \mathrm{A}_1 + \mathrm{D}_1\mathrm{A}_2^T & \mathrm{A}_2 + \mathrm{D}_1\mathrm{A}_4 \end{array}\right)$$

for some $(n-k) \times k$ matrix D_1, determined by π.

Let $\{|x\rangle \mid x \in \mathbb{Z}/D\mathbb{Z}\}$ be a basis for $\mathbb{C}^D$.

Let η be a primitive D-th root of unity in $\mathbb{C}$.

For $a, b \in \mathbb{Z}/D\mathbb{Z}$, define Weyl linear operators $X(a)$ and $Z(b)$ as

$$X(a)\,|x\rangle = |x + a\rangle$$

and

$$Z(b)\,|x\rangle = \eta^{bx}\,|x\rangle$$

$X(a)Z(b)$ form a basis for the endomorphisms of $\mathbb{C}^D$.

A Hilbert space is a complex vector space

$$\mathbb{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2} \otimes \cdots \otimes \mathbb{C}^{D_n}$$

the elements of which are

$$|\psi\rangle = \sum_{x \in \prod_{i=1}^{n} \mathbb{Z}/D_i\mathbb{Z}} c_x \, |x_1 x_2 \cdots x_n\rangle$$

The real number $c_x \overline{c_x}$ is the probability of finding the unit vector $|\psi\rangle$ in the quantum state in $|x_1 x_2 \cdots x_n\rangle$.

Example

The Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The linear maps on $\mathbb{H}$

$$\sigma_1 \otimes \cdots \otimes \sigma_n,$$

where the Weyl operator $\sigma_i = X(a_i)Z(b_i)$, form a basis for the linear maps on this space.

Let Q be a subspace of $\mathbb{H}$. The set of Weyl operators $\mathcal{E}$ is a correctable set of errors for Q if for all $E, E' \in \mathcal{E}$ and orthogonal $|\phi\rangle, |\phi'\rangle \in Q$,

$E|\phi\rangle$ and $E'|\phi'\rangle$ are orthogonal.

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. The "commuting" relation is

$$X(a)Z(b)X(a')Z(b') = \eta^{a \cdot b' - b \cdot a'} X(a')Z(b')X(a)Z(b)$$

The (multiplicative) subgroup

$$S = \{X(a)Z(b) \mid (a,b) \in C\}$$

is commutative iff $C \subseteq \mathbb{F}_p^{2n}$ is a totally isotropic subspace w.r.t the symplectic form.

Let

$$Q(S) = \{|\phi\rangle \in (\mathbb{C}^p)^{\otimes n} \mid M|\phi\rangle = |\phi\rangle, \forall M \in S\}$$

$Q(S)$ can correct Weyl all errors of weight at most $(d-1)/2$ (weight=non-identity components) where d is minimal such that there are d dependent points on distinct lines of $\mathcal{X}$.

Entanglement

The Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is absolutely maximally entangled state of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Taking the partial trace of the rank 1 matrix on either component

$$\frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

we get

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathrm{I}_2.$$

If $k = 0$ and $d = \lfloor \frac{1}{2}(n+2) \rfloor$ then a unit vector in the one-dim subspace $Q(S)$ is an absolutely maximally entangled state.

From the regular hyperoval $n = 6$, $d = 4$ (no three dependent points on the distinct lines since any three lines span $\mathrm{PG}(5,2)$, gives an absolutely maximally entangled state of

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$

The partial trace on any three subsystems gives the identity on

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$

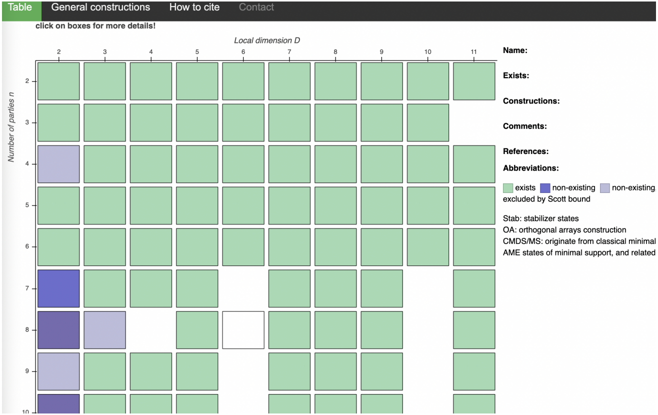
This is a regular hyperoval in PG(2, 4).

$$\mathrm{H} = \left(\begin{array}{cccccc||cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right).$$

$$|\phi\rangle = \frac{1}{\sqrt{8}}(|000000\rangle + |100000\rangle + |010000\rangle - |110000\rangle +$$

60 other terms.

Huber's table on existence of absolutely max. entangled states on $(\mathbb{C}^D)^{\otimes n}$



[Ball, Moreno, Simoens (2025)] [IEEE Trans.]
There is no stabilised absolutely maximally entangled state in $(\mathbb{C}^4)^{\otimes 8}$.

Some additional results on existence of absolutely maximally entangled states.

[Higuchi, Sudbery 2000] [Phys. Lett. A]
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, no.

(A stabilised state is equivalent to partial spread of 4 lines $\mathcal{X}$ in PG(3,2) for which C is symmetric matrix.)

[F. Huber, C. Eltschka, J. Siewert and O. Gühne 2018] [J. Phys. A]
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, no.
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, no.
$\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, yes.

[Ball, Zhang 2025] [arxiv next week maybe]
$\mathbb{C}^r \otimes \mathbb{C}^q \otimes \mathbb{C}^q \otimes \mathbb{C}^q$, yes.
for all $r \leqslant q-1$ and q a prime power.