The Kovalevski Configuration of a Quartic Curve over a Finite Field

Anton Betten, September 2025 — Irsee

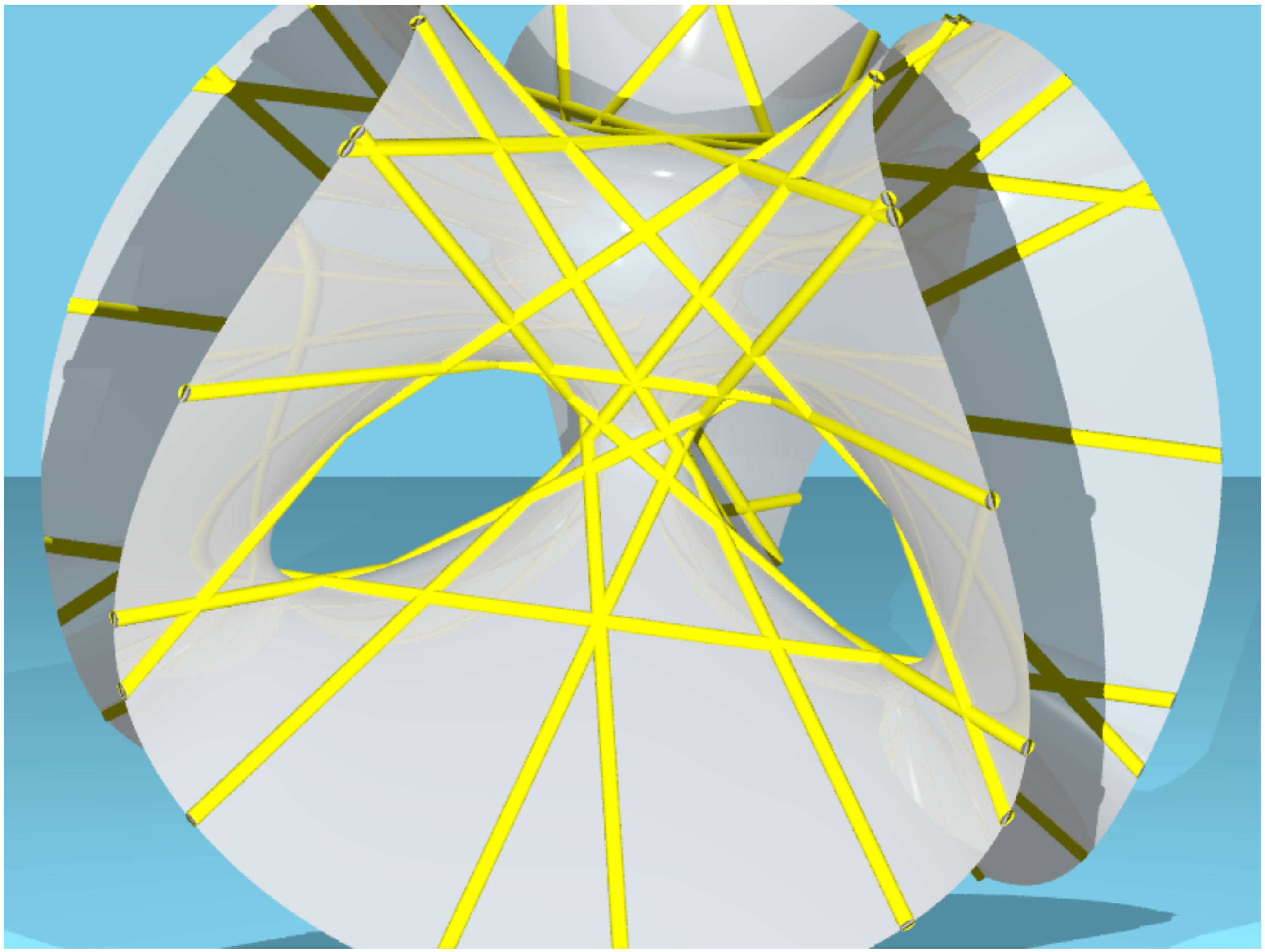
Outline

- Cubic Surfaces
- Quartic Curves
- Classification by Computer
- Normal Form
- Examples: K=63, K=21, K=15.
- Relation between K and E
- Relation between the group of the surface and the group of the quartic curve.

Cubic Surfaces

- Algebraic Varieties
- A cubic polynomial in 4 variables defines a cubic surface in P^3.
- We want the surface to be smooth -> 27 lines
- We want the field to be finite: Fq -> 27 lines or less.
- We want to classify all surfaces up to isomorphism.
- Previous work: Betten/Karaoglu 2019 (27 lines, q <= 97).

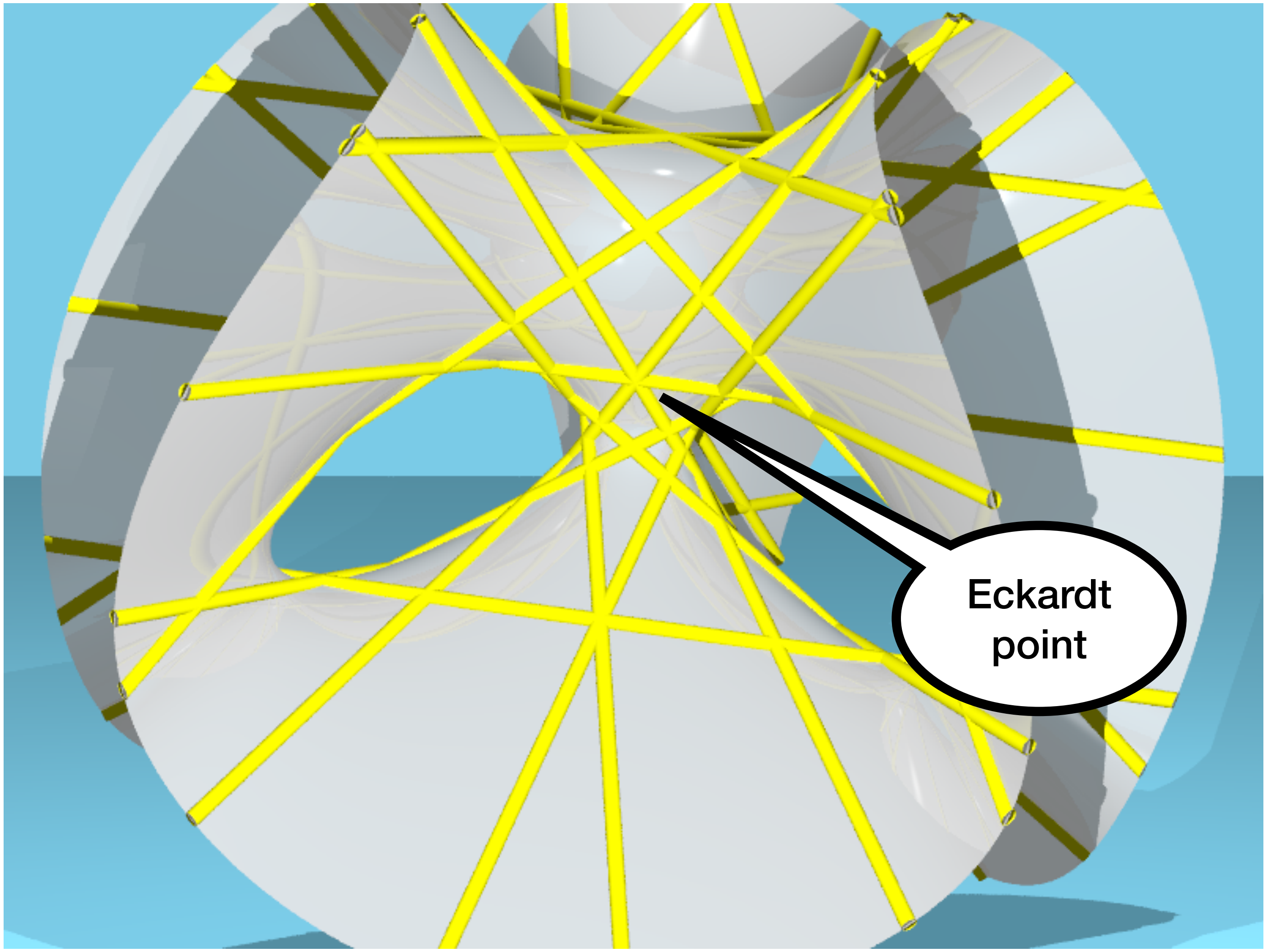
The Clebsch Cubic Surface



Cubic Surfaces

- Geometric Invariants:
- An Eckardt point is a point on the surface where three lines are concurrent (first studied by Eckardt in 1876).

The Clebsch Cubic Surface



Schläfli Double Six

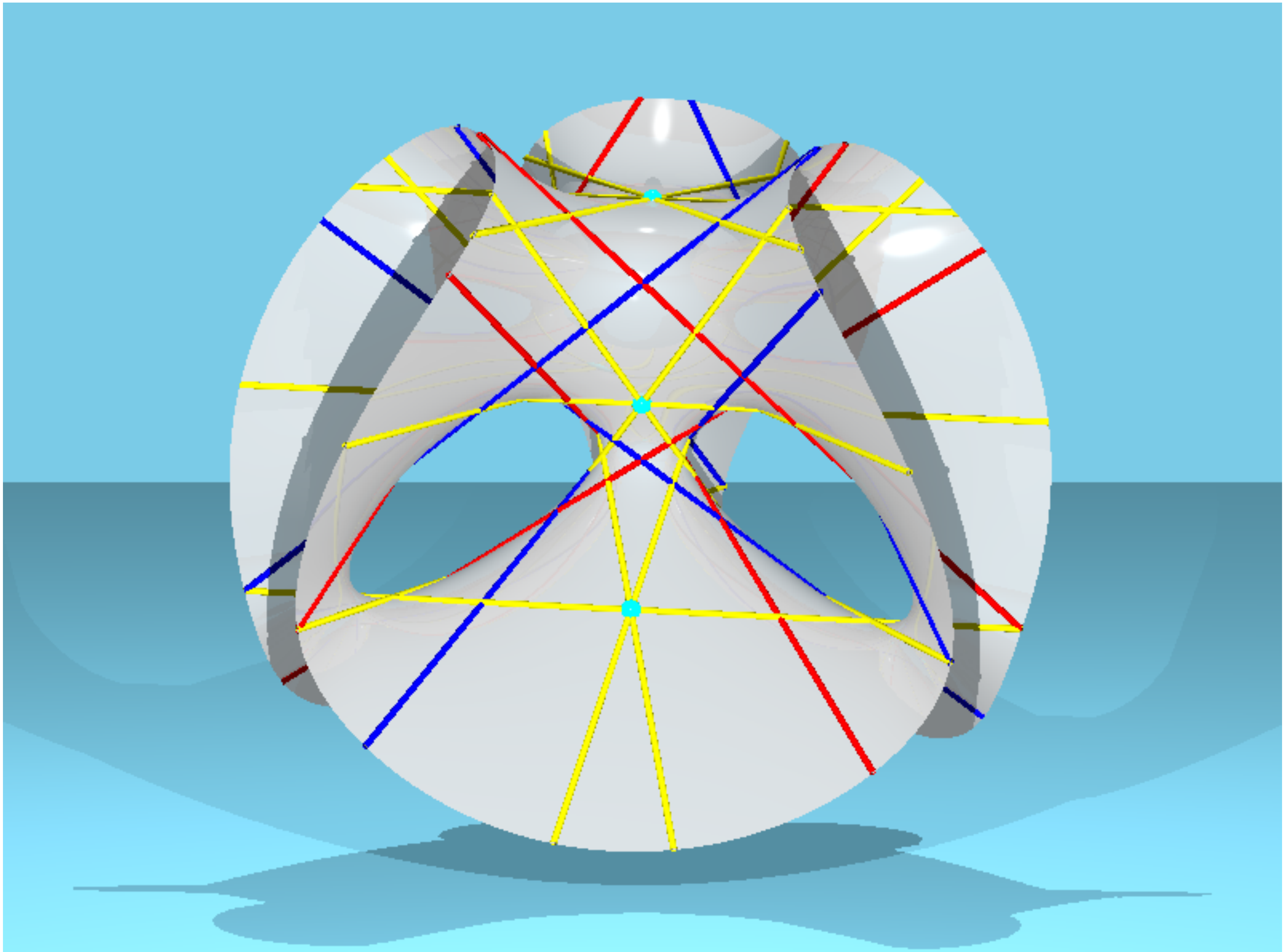
- A Schläfli double six is two sets of six lines each, say a1,…,a6, b1,…,b6 such that ai and bj intersect if and only if i <> j.

- Notation:

a1	a2	a3	a4	a5	a6
b1	b2	b3	b4	b5	b6

- A Schläfli double six determines a unique cubic surface with 27 lines. Conversely, a cubic surface with 27 lines determines 36 double sixes.

Eckardt Points



Properties

- The ai lines are shown in red
- The bj lines are shown in blue
- There are 15 further lines, shown in yellow.
- These are the cij-lines, where $c_{ij} = a_i b_j \cap a_j b_i$.
- The Eckardt points are shown in Turquoise.
- There are at most 45 Eckardt points.
- Eckardt points are related to tritangent planes.

Classification

- Classification by substructure or related object:
- Identify a related object that is easier to classify.
- Then lift the classification from one class of objects to the other.
- Example: Cubic surface -> double six
- double six -> five-plus-one
- Cubic surface -> six-arc
- Quartic curve -> cubic surface

Classification of Combinatorial Objects

- A combinatorial object is an object for which the isomorphism question can be settled by using isomorphism of 0,1-matrices.
- Example: designs, graphs, sets in projective space
- Not a combinatorial object: algebraic varieties.

Canonical Forms

- Any class of combinatorial object can be classified by using canonical forms.
- Canonical forms can be computed by reducing canonical forms of 0,1-matrices to canonical forms of bipartite graphs.
- Canonical forms of graphs (possibly with vertex partitions) can be done by using graph theoretic tools like Brendan McKay’s software Nauty.
- Canonical forms do not work well for cubic surfaces, as the required graphs are simply too big. So, using related objects is better.

Quartic Curves

- Smooth Quartic curves are related to smooth cubic surfaces.

- Given a cubic surface with 27 lines, and a point P not on any line, we can project along the polar cone onto a plane to get a quartic curve.

- The 27 lines of the surface become bitangents. The tangent plane at P becomes a bitangent as well. This gives all 28 bitangents.

- We can use a hybrid method to classify quartic curves.

Quartic Curves

- Let q be a prime power.
- Classify all smooth cubic surfaces over Fq.
- For each surface, and for each point not on any line, form the associated quartic curve.
- Let us call a pair of a cubic surface and a quartic curve related if there is a point P on the surfaces s.t. the projection along the polar cone results in the quartic curve.
- To perform the classification, we need the flag orbits.
- So, for a given isomorphism type of cubic surface, we may limit us to consider only the orbits on points not on lines.
- Once done, we identify the isomorphism types of quartic curves from these flag orbits by using canonical forms (Nauty).

Results

- Together with my coauthors (Alazemi and Karaoglu) we were able to extend the classification of quartic curves from q=19 to all q <= 49.
- This raises an issue: how to present the objects in a form that is reasonably compact?
- Idea: use Normal forms.

Normal Forms

- A normal form is a description involving parameters such that every object of a given type has a description in the given form, up to isomorphism.

- Example:

- In 2022, Betten and Karaoglu give a normal form for cubic surfaces with 27 lines.

- Previous normal forms were given by Cayley (in the 19th century) and other researchers.

- But: not all of these normal forms work over finite fields.

Normal Form (B+Karaoglu 2022)

F_a,b,c,d normal form for any cubic surface (with 27 lines)

```
F_abcd_eqn="-(a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*(b - d)*X0^2*X2 \
+ (a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*(a + b - c - d)*X0*X1*X2 \
+ (a^2*c - a^2*d - a*c^2 + b*c^2 + a*d - b*c)*(b - d)*X0*X1*X3 \
- (a*d - b*c)*(a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*X0*X2^2 \
- (a^2*c*d - a*b*c^2 - a^2*d + a*b*d + b*c^2 - b*c*d)*(b - d)*X0*X2*X3 \
- (a - c)*(a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*X1^2*X2 \
- (a - c)*(a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*X1^2*X3 \
+ (a*d - b*c)*(a*b*c - a*b*d - a*c*d + b*c*d + a*d - b*c)*X1*X2^2 \
+ ((1+1)*a^2*b*c*d - a^2*b*d^2 - (1+1)*a^2*c*d^2 \
- (1+1)*a*b^2*c^2 + a*b^2*c*d + (1+1)*a*b*c^2*d + a*b*c*d^2 \
- b^2*c^2*d - a^2*b*c + a^2*c*d + a^2*d^2 + a*b^2*c + a*b*c^2 \
- (1+1+1+1)*a*b*c*d - a*c^2*d + a*c*d^2 + b^2*c^2)*X1*X2*X3 \
+ c*a*(a*d - b*c - a + b + c - d)*(b - d)*X1*X3^2"
```

Normal Forms (New)

Quartic curve with 28 bitangents:

Theorem 1 *Any smooth quartic curve over a field* $\mathbb{F}$ *has a representation as* $\mathcal{C}_{a,b,c,d,e,f}(x_1, x_2, x_3)$ *with* $a, b, c, d, e, f \in \mathbb{F}$*, where* $\mathcal{C}_{a,b,c,d,e,f}$ *is given as*

$$\sum_{i \leq j \leq k \leq l} c_{ijkl}\, x_i x_j x_k x_l$$

where…

$$c_{1111} = (a-c)^2(abc - abd - acd + bcd + ad - bc)^2(p_2 + p_3)^2$$

$$
\begin{aligned}
c_{1112} \quad = \quad & 4((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0p_2 \\
& +(a^2c-a^2d-ac^2+bc^2+ad-bc)(b-d)p_0p_3 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1p_2 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1p_3 \\
& +(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2^2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d \\
& +abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_2p_3 \\
& +ca(ad-bc-a+b+c-d)(b-d)p_3^2)(a-c)(abc-abd-acd+bcd+ad-bc) \\
& +2((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd \\
& +2abc^2d+abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2 \\
& +ab^2c+abc^2-4abcd-ac^2d+acd^2 \\
& +b^2c^2)p_3)(-(a-c)(abc-abd-acd+bcd+ad-bc)p_2 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1113} \quad = \quad & 4((abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0p_2 \\
& +(a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b - d)p_0p_3 \\
& -2(a - c)(abc - abd - acd + bcd + ad - bc)p_1p_2 \\
& -2(a - c)(abc - abd - acd + bcd + ad - bc)p_1p_3 \\
& +(ad - bc)(abc - abd - acd + bcd + ad - bc)p_2^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d \\
& +abcd^2 - b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 \\
& -4abcd - ac^2d + acd^2 + b^2c^2)p_2p_3 \\
& +ca(ad - bc - a + b + c - d)(b - d)p_3^2)(a - c)(abc - abd - acd + bcd + ad - bc) \\
& +2((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b - d)p_0 \\
& -2(a - c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2 \\
& +2ca(ad - bc - a + b + c - d)(b - d)p_3)(-(a - c)(abc - abd - acd + bcd + ad - bc)p_2 \\
& -(a - c)(abc - abd - acd + bcd + ad - bc)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1122} \quad = \quad & 4(-(abc-abd-acd+bcd+ad-bc)(b-d)p_0^2 \\
& +(abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0p_1 \\
& -2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_0p_2 \\
& -(a^2cd-abc^2-a^2d+abd+bc^2-bcd)(b-d)p_0p_3 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_1^2 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_1p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d \\
& +abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2 \\
& -4abcd-ac^2d+acd^2+b^2c^2)p_1p_3) \\
& (a-c)(abc-abd-acd+bcd+ad-bc) \\
& -4((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0p_2 \\
& +(a^2c-a^2d-ac^2+bc^2+ad-bc)(b-d)p_0p_3 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1p_2 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1p_3 \\
& +(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2^2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd \\
& +2abc^2d+abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2 \\
& +ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_2p_3 \\
& +ca(ad-bc-a+b+c-d)(b-d)p_3^2)(ad-bc) \\
& (abc-abd-acd+bcd+ad-bc) \\
& +2(-(ad-bc)(abc-abd-acd+bcd+ad-bc)p_0 \\
& +(ad-bc)(abc-abd-acd+bcd+ad-bc)p_1) \\
& (-(a-c)(abc-abd-acd+bcd+ad-bc)p_2 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_3) \\
& +((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d \\
& +abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c \\
& +abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_3)^2
\end{aligned}
$$

$$
\begin{aligned}
c_{1123} \;=\; & 4((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_1 \\
& -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_2 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_1p_3)(a-c)(abc - abd - acd + bcd + ad - bc) \\
& +4(-(abc - abd - acd + bcd + ad - bc)(b-d)p_0^2 \\
& +(abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0p_1 \\
& -2(ad - bc)(abc - abd - acd + bcd + ad - bc)p_0p_2 \\
& -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_3 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
& +2(ad - bc)(abc - abd - acd + bcd + ad - bc)p_1p_2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_3) \\
& (a-c)(abc - abd - acd + bcd + ad - bc) \\
& -4((abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0p_2 \\
& +(a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_3 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1p_2 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1p_3 \\
& +(ad - bc)(abc - abd - acd + bcd + ad - bc)p_2^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2p_3 \\
& +ca(ad - bc - a + b + c - d)(b-d)p_3^2)(2a^2bcd - a^2bd^2 - 2a^2cd^2 \\
& -2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d - a^2bc + a^2cd \\
& +a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2) \\
& +2(-(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d \\
& +acd^2 + b^2c^2)p_1)(-(a-c)(abc - abd - acd + bcd + ad - bc)p_2 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_3) \\
& +2((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0 - 2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_3)((abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +2(ad - bc)(abc - abd - acd + bcd + ad - bc)p_2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1133} \quad = \quad & 4((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_1 \\
& -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_2 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_1p_3) \\
& (a-c)(abc - abd - acd + bcd + ad - bc) \\
& -4((abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0p_2 \\
& +(a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_3 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1p_2 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1p_3 \\
& +(ad - bc)(abc - abd - acd + bcd + ad - bc)p_2^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2p_3 \\
& +ca(ad - bc - a + b + c - d)(b-d)p_3^2)ca(ad - bc - a + b + c - d)(b-d) \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_1 \\
& (-(a-c)(abc - abd - acd + bcd + ad - bc)p_2 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_3) \\
& +((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_3)^2
\end{aligned}
$$

$$
\begin{aligned}
c_{1222} \quad = \quad & -4(-(abc-abd-acd+bcd+ad-bc)(b-d)p_0^2 \\
& +(abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0p_1 \\
& -2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_0p_2 \\
& -(a^2cd-abc^2-a^2d+abd+bc^2-bcd)(b-d)p_0p_3 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_1^2 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_1p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2 \\
& -b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_1p_3) \\
& (ad-bc)(abc-abd-acd+bcd+ad-bc) \\
& +2(-(ad-bc)(abc-abd-acd+bcd+ad-bc)p_0 \\
& +(ad-bc)(abc-abd-acd+bcd+ad-bc)p_1) \\
& ((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1+2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2 \\
& -b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1223} \quad = \quad & -4((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_1 \\
& -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_2 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_1p_3)(ad-bc)(abc - abd - acd + bcd + ad - bc) \\
& -4(-(abc - abd - acd + bcd + ad - bc)(b-d)p_0^2 \\
& +(abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0p_1 \\
& -2(ad-bc)(abc - abd - acd + bcd + ad - bc)p_0p_2 \\
& -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_3 \\
& -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
& +2(ad-bc)(abc - abd - acd + bcd + ad - bc)p_1p_2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d - a^2bc \\
& +a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_3) \\
& (2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
& -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2) \\
& +2(-(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1) \\
& ((abc - abd - acd + bcd + ad - bc)(a + b - c - d)p_0 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +2(ad-bc)(abc - abd - acd + bcd + ad - bc)p_2 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_3) \\
& +2(-(ad-bc)(abc - abd - acd + bcd + ad - bc)p_0 \\
& +(ad-bc)(abc - abd - acd + bcd + ad - bc)p_1)((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0 \\
& -2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
& +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
& -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2 \\
& +2ca(ad - bc - a + b + c - d)(b-d)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1233} \quad = \quad & -4((a^2c-a^2d-ac^2+bc^2+ad-bc)(b-d)p_0p_1 \\
& -(a^2cd-abc^2-a^2d+abd+bc^2-bcd)(b-d)p_0p_2 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_1^2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2-b^2c^2d \\
& -a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_1p_2 \\
& +2ca(ad-bc-a+b+c-d)(b-d)p_1p_3) \\
& (2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2 \\
& -b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2) \\
& -4(-(abc-abd-acd+bcd+ad-bc)(b-d)p_0^2 \\
& +(abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0p_1 \\
& -2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_0p_2 \\
& -(a^2cd-abc^2-a^2d+abd+bc^2-bcd)(b-d)p_0p_3 \\
& -(a-c)(abc-abd-acd+bcd+ad-bc)p_1^2 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_1p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d \\
& +abcd^2-b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_1p_3) \\
& ca(ad-bc-a+b+c-d)(b-d)+2ca(ad-bc-a+b+c-d)(b-d)p_1 \\
& ((abc-abd-acd+bcd+ad-bc)(a+b-c-d)p_0 \\
& -2(a-c)(abc-abd-acd+bcd+ad-bc)p_1 \\
& +2(ad-bc)(abc-abd-acd+bcd+ad-bc)p_2 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2-b^2c^2d \\
& -a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_3) \\
& +2(-(a^2cd-abc^2-a^2d+abd+bc^2-bcd)(b-d)p_0 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2-b^2c^2d \\
& -a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_1) \\
& ((a^2c-a^2d-ac^2+bc^2+ad-bc)(b-d)p_0-2(a-c)(abc-abd-acd+bcd+ad-bc)p_1 \\
& +(2a^2bcd-a^2bd^2-2a^2cd^2-2ab^2c^2+ab^2cd+2abc^2d+abcd^2 \\
& -b^2c^2d-a^2bc+a^2cd+a^2d^2+ab^2c+abc^2-4abcd-ac^2d+acd^2+b^2c^2)p_2 \\
& +2ca(ad-bc-a+b+c-d)(b-d)p_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{1333} &= -4((a^2c - a^2d - ac^2 + bc^2 + ad - bc)(b-d)p_0p_1 \\
&\quad -(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0p_2 \\
&\quad -(a-c)(abc - abd - acd + bcd + ad - bc)p_1^2 \\
&\quad +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
&\quad -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1p_2 \\
&\quad +2ca(ad - bc - a + b + c - d)(b-d)p_1p_3)ca(ad - bc - a + b + c - d)(b-d) \\
&\quad +2ca(ad - bc - a + b + c - d)(b-d)p_1((a^2c - a^2d - ac^2 + bc^2 + ad - bc) \\
&\quad (b-d)p_0 - 2(a-c)(abc - abd - acd + bcd + ad - bc)p_1 \\
&\quad +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
&\quad -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_2 \\
&\quad +2ca(ad - bc - a + b + c - d)(b-d)p_3) \\
c_{2222} &= (ad-bc)^2(abc - abd - acd + bcd + ad - bc)^2(p_0 - p_1)^2 \\
c_{2223} &= 2(-(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0 \\
&\quad +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d - a^2bc \\
&\quad +a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1) \\
&\quad (-(ad-bc)(abc - abd - acd + bcd + ad - bc)p_0 \\
&\quad +(ad-bc)(abc - abd - acd + bcd + ad - bc)p_1) \\
c_{2233} &= 2ca(ad - bc - a + b + c - d)(b-d)p_1(-(ad-bc)(abc - abd - acd + bcd + ad - bc)p_0 \\
&\quad +(ad-bc)(abc - abd - acd + bcd + ad - bc)p_1) \\
&\quad +(-(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0 \\
&\quad +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 - b^2c^2d \\
&\quad -a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1)^2 \\
c_{2333} &= 2ca(ad - bc - a + b + c - d)(b-d)p_1 \\
&\quad (-(a^2cd - abc^2 - a^2d + abd + bc^2 - bcd)(b-d)p_0 \\
&\quad +(2a^2bcd - a^2bd^2 - 2a^2cd^2 - 2ab^2c^2 + ab^2cd + 2abc^2d + abcd^2 \\
&\quad -b^2c^2d - a^2bc + a^2cd + a^2d^2 + ab^2c + abc^2 - 4abcd - ac^2d + acd^2 + b^2c^2)p_1) \\
c_{3333} &= c^2a^2(ad - bc - a + b + c - d)^2(b-d)^2p_1^2
\end{aligned}
$$

with…

$$
\begin{aligned}
p_0 &= bd(a-c)(ad-bc-a+b+c-d)y_0^2y_2^2 \\
&\quad -(a-c)(ad^2-b^2c+b^2d-bd^2-ad+bc)y_0^2y_1y_2 \\
&\quad -(a-c)(abc-abd-acd+bcd+ad-bc)y_0y_1^2y_2 \\
&\quad +(a-c)(-abd^2+b^2cd+abc-acd+ad^2-b^2c)y_0y_1y_2^2 \\
p_1 &= -ac(b-d)(ad-bc-a+b+c-d)y_1^2y_2^2 \\
&\quad +(b-d)(abc-abd-acd+bcd+ad-bc)y_0^2y_1y_2 \\
&\quad -(b-d)(a^2c-a^2d-ac^2+bc^2+ad-bc)y_0y_1^2y_2 \\
&\quad +(b-d)(a^2cd-abc^2-a^2d+abd+bc^2-bcd)y_0y_1y_2^2 \\
p_2 &= (b-d)(a^2cd-abc^2-a^2d+abd+bc^2-bcd)y_0y_2^3 \\
&\quad -ac(b-d)(ad-bc-a+b+c-d)y_1y_2^3 \\
&\quad +(b-d)(abc-abd-acd+bcd+ad-bc)y_0^2y_2^2 \\
&\quad -(b-d)(a^2c-a^2d-ac^2+bc^2+ad-bc)y_0y_1y_2^2 \\
p_3 &= -(abc-abd-acd+bcd+ad-bc)(ad-bc)y_0y_2^3 \\
&\quad +(abc-abd-acd+bcd+ad-bc)(ad-bc)y_1y_2^3 \\
&\quad -(b-d)(abc-abd-acd+bcd+ad-bc)y_0^2y_2^2 \\
&\quad -(a-c)(abc-abd-acd+bcd+ad-bc)y_1^2y_2^2 \\
&\quad +(abc-abd-acd+bcd+ad-bc)(a+b-c-d)y_0y_1y_2^2,
\end{aligned}
$$

with

$$y_0 = e,\ y_1 = f,\ y_2 = 1.$$

Proof: 3 pages.

Classification

- B + Karaoglu (2019) classify all cubic surfaces with 27 lines over finite fields Fq with q <= 97.
- New: classification of quartic curves with 28 bitangents over finite fields Fq with q <= 49 (for q <= 19, the classification was known before).
- Method: related structures + canonical forms.
- Verification using Kaplan's formula.

Kaplan 2013: The total number of quartic curves with 28 bitangents (not up to isomorphism) over $\mathbb{F}_q$ is equal to

$$\frac{2(q-7)(q-5)(q-3)(q^3-20q^2+119q-175)}{2903040}|\mathrm{P\Gamma L}(3,q)|$$

Quartic Curves

q=9,11,13,17,19

(previously known

due to work of Kaplan, Knecht)

q	Iso	abcdef	$G_{\mathcal{C}}$
9	0	6,3,4,6,2,5	12096
11	0	8,5,3,2,10,7	168
13	0	3,2,10,4,2,0	24
13	1	3,2,10,4,8,0	48
17	0	2,7,16,3,15,8	4
17	1	2,7,16,3,14,13	2
17	2	2,7,16,3,9,4	96
17	3	10,16,4,5,6,12	24
17	4	10,16,4,5,14,6	6
17	5	10,16,4,5,8,2	8
17	6	3,11,2,7,8,12	24
19	0	2,14,15,18,13,6	2
19	1	2,14,15,18,7,8	6
19	2	2,14,15,18,16,5	8
19	3	2,14,15,18,17,11	2
19	4	2,14,15,18,13,16	2
19	5	2,14,15,18,14,17	9
19	6	2,14,15,18,7,10	8
19	7	2,14,15,18,4,12	4
19	8	2,14,15,18,8,9	8
19	9	2,14,15,18,18,11	6
19	10	2,14,15,18,11,7	2
19	11	2,14,15,18,9,15	24
19	12	2,14,15,18,6,7	24
19	13	11,13,12,15,13,7	24

Quartic Curves

q=23

(new)

q	Iso	abcdef	$G_{\mathcal{C}}$	q	Iso	abcdef	$G_{\mathcal{C}}$
23	0	2,9,8,19,9,16	1	23	21	2,9,8,19,18,6	2
23	1	2,9,8,19,12,7	2	23	22	2,9,8,19,11,16	2
23	2	2,9,8,19,11,10	4	23	23	2,9,8,19,17,16	24
23	3	2,9,8,19,16,18	6	23	24	2,9,8,19,20,12	4
23	4	2,9,8,19,21,3	2	23	25	2,9,8,19,20,14	8
23	5	2,9,8,19,3,11	2	23	26	2,9,8,19,14,3	8
23	6	2,9,8,19,7,22	6	23	27	13,4,19,18,15,11	2
23	7	2,9,8,19,17,15	2	23	28	13,4,19,18,11,16	2
23	8	2,9,8,19,10,13	1	23	29	13,4,19,18,5,3	4
23	9	2,9,8,19,19,11	4	23	30	14,7,18,14,10,15	24
23	10	2,9,8,19,15,16	1	23	31	14,7,18,14,11,12	8
23	11	2,9,8,19,20,4	1	23	32	3,21,8,12,10,7	4
23	12	2,9,8,19,13,7	6	23	33	3,21,8,12,4,11	24
23	13	2,9,8,19,6,10	2	23	34	3,21,8,12,5,13	2
23	14	2,9,8,19,11,13	2	23	35	3,21,8,12,9,15	8
23	15	2,9,8,19,20,18	2	23	36	3,21,8,12,15,19	6
23	16	2,9,8,19,5,2	1	23	37	3,21,8,12,10,4	6
23	17	2,9,8,19,17,20	1	23	38	3,21,8,12,9,8	8
23	18	2,9,8,19,11,3	2	23	39	17,10,9,11,11,8	8
23	19	2,9,8,19,10,16	2	23	40	4,20,3,4,9,14	168
23	20	2,9,8,19,14,2	2				

Quartic Curves

q=25

(new)

q	Iso	abcdef	$G_{\mathcal{C}}$
25	0	14,19,21,5,5,23	6
25	1	14,19,21,5,8,12	4
25	2	14,19,21,5,2,21	16
25	3	14,19,21,5,20,14	1
25	4	14,19,21,5,10,15	4
25	5	14,19,21,5,18,10	2
25	6	14,19,21,5,2,13	4
25	7	14,19,21,5,20,3	1
25	8	14,19,21,5,7,23	1
25	9	14,19,21,5,18,11	1
25	10	14,19,21,5,3,22	2
25	11	14,19,21,5,22,12	4
25	12	14,19,21,5,6,16	8
25	13	14,19,21,5,13,8	2
25	14	14,19,21,5,5,10	2
25	15	14,19,21,5,8,22	2
25	16	14,19,21,5,17,2	2
25	17	14,19,21,5,18,21	2
25	18	14,19,21,5,11,6	1
25	19	14,19,21,5,2,8	2
25	20	14,19,21,5,6,17	2
25	21	14,19,21,5,5,9	2
25	22	12,17,8,18,7,2	12
25	23	12,17,8,18,14,9	2
25	24	12,17,8,18,11,14	12
25	25	12,17,8,18,20,6	2
25	26	12,17,8,18,24,12	16
25	27	20,12,21,5,8,24	2
25	28	20,12,21,5,8,16	192
25	29	9,8,20,21,10,14	2
25	30	9,8,20,21,22,17	2
25	31	9,8,20,21,2,10	2
25	32	9,8,20,21,18,7	4
25	33	9,8,20,21,19,22	4
25	34	9,8,20,21,13,7	4
25	35	9,8,20,21,8,3	6
25	36	14,19,10,6,11,16	12
25	37	14,19,10,6,24,2	16
25	38	14,19,10,6,20,9	16
25	39	14,19,10,6,11,3	8
25	40	23,22,9,20,22,8	6
25	41	23,22,9,20,10,24	8
25	42	10,3,12,20,2,21	96
25	43	10,3,12,20,5,17	48
25	44	2,23,16,9,7,2	336

Quartic Curves

q=27

(new)

q	Iso	abcdef	$G_{\mathcal{C}}$
27	0	5,18,13,15,4,21	1
27	1	5,18,13,15,2,17	1
27	2	5,18,13,15,16,13	1
27	3	5,18,13,15,10,11	2
27	4	5,18,13,15,9,20	2
27	5	5,18,13,15,24,16	1
27	6	5,18,13,15,24,19	8
27	7	5,18,13,15,16,14	2
27	8	5,18,13,15,7,26	2
27	9	5,18,13,15,6,16	2
27	10	5,18,13,15,8,21	4
27	11	5,18,13,15,16,20	2
27	12	5,18,13,15,22,26	6
27	13	5,18,13,15,24,25	2
27	14	5,18,13,15,12,25	1
27	15	5,18,13,15,9,4	6
27	16	5,18,13,15,3,20	6
27	17	5,18,13,15,19,13	2
27	18	5,18,13,15,25,4	4
27	19	5,18,13,15,10,9	2
27	20	5,18,13,15,18,17	1

q	Iso	abcdef	$G_{\mathcal{C}}$
27	21	22,8,26,4,12,3	1
27	22	22,8,26,4,14,12	1
27	23	22,8,26,4,11,17	2
27	24	22,8,26,4,13,21	2
27	25	22,8,26,4,14,10	4
27	26	22,8,26,4,11,25	24
27	27	22,8,26,4,20,7	2
27	28	22,8,26,4,24,26	1
27	29	22,8,26,4,2,10	6
27	30	22,8,26,4,3,19	1
27	31	22,8,26,4,23,26	8
27	32	22,8,26,4,3,22	2
27	33	26,13,3,23,6,19	2
27	34	26,13,3,23,5,25	9
27	35	26,13,3,23,16,20	2
27	36	26,13,3,23,16,21	8
27	37	5,15,13,20,15,8	162
27	38	5,15,13,20,23,7	4
27	39	16,11,20,13,10,16	24

Verification

q	nb	sum of inverse aut-group orders	factor
9	1	$\frac{1}{12096}$	$\frac{1}{12096}$
11	1	$\frac{1}{168}$	$\frac{1}{168}$
13	2	$\frac{1}{24}+\frac{1}{48}$	$\frac{1}{16}$
17	7	$\frac{1}{4}+\frac{1}{2}+\frac{1}{96}+\frac{2}{24}+\frac{1}{6}+\frac{1}{8}$	$\frac{109}{96}$
19	14	$\frac{4}{2}+\frac{2}{6}+\frac{3}{8}+\frac{1}{9}+\frac{1}{4}+\frac{3}{24}$	$\frac{115}{36}$
23	41	$\frac{6}{1}+\frac{15}{2}+\frac{5}{4}+\frac{5}{6}+\frac{3}{24}+\frac{6}{8}+\frac{1}{168}$	$\frac{461}{28}$
25	45	$\frac{3}{6}+\frac{7}{4}+\frac{4}{16}+\frac{5}{1}+\frac{16}{2}+\frac{3}{8}+\frac{3}{12}+\frac{1}{192}+\frac{1}{96}+\frac{1}{48}+\frac{1}{336}$	$\frac{21725}{1344}$
27	40	$\frac{10}{1}+\frac{15}{2}+\frac{3}{8}+\frac{4}{4}+\frac{4}{6}+\frac{2}{24}+\frac{1}{9}+\frac{1}{162}$	$\frac{12793}{648}$
29	175	$\frac{60}{1}+\frac{69}{2}+\frac{4}{24}+\frac{11}{6}+\frac{19}{4}+\frac{10}{8}+\frac{1}{16}+\frac{1}{168}$	$\frac{34463}{336}$
31	270	$\frac{106}{2}+\frac{106}{1}+\frac{15}{8}+\frac{23}{4}+\frac{12}{6}+\frac{6}{24}+\frac{2}{3}$	$\frac{4096}{24}$
37	845	$\frac{458}{1}+\frac{21}{6}+\frac{279}{2}+\frac{53}{4}+\frac{5}{24}+\frac{22}{8}+\frac{3}{3}+\frac{1}{16}+\frac{1}{48}+\frac{1}{9}+\frac{1}{168}$	$\frac{155839}{252}$
41	1637	$\frac{80}{4}+\frac{477}{2}+\frac{1015}{1}+\frac{8}{24}+\frac{28}{8}+\frac{27}{6}+\frac{1}{96}+\frac{1}{16}$	$\frac{41021}{32}$
43	2234	$\frac{1449}{1}+\frac{611}{2}+\frac{37}{8}+\frac{93}{4}+\frac{7}{24}+\frac{31}{6}+\frac{5}{3}+\frac{1}{168}$	$\frac{300637}{168}$
47	3969	$\frac{2792}{1}+\frac{952}{2}+\frac{132}{4}+\frac{38}{6}+\frac{11}{24}+\frac{44}{8}$	$\frac{79519}{24}$
49	2701	$\frac{679}{2}+\frac{17}{6}+\frac{1827}{1}+\frac{114}{4}+\frac{10}{12}+\frac{10}{16}+\frac{32}{8}+\frac{1}{192}+\frac{3}{3}+\frac{3}{48}+\frac{3}{24}+\frac{1}{32}+\frac{1}{96}$	$\frac{423269}{192}$

Kaplan:

# of quartic curves =

$$\boxed{\frac{2(q-7)(q-5)(q-3)(q^3-20q^2+119q-175)}{2903040}}|\mathrm{P\Gamma L}(3,q)|$$

factor

How to unpack the data?

- How can we see the curves?
- Get the parameters a,b,c,d,e,f from the table and plug into the given formulae.
- Example: For $q=9$, we find a,b,c,d,e,f=6,3,4,6,2,5.
- How do we read this?

Example 2 *We use the polynomial $X^2 + X + 2$ to create the finite field $\mathbb{F}_9$ as an extension of degree two of $\mathbb{F}_3$. The elements of $\mathbb{F}_9$ are coded as integers in the interval from 0 to 8. We use the representation of elements as $\omega = a_1X + a_0$ with $0 \leq a_0, a_1 < 3$ and code ω as $3a_1 + a_0$. This way, we have field elements 0, 1, 2, $3 = \omega$, $4 = \omega + 1$, $5 = \omega + 2$, $6 = 2\omega$, $7 = 2\omega + 1$ and $8 = 2\omega + 2$. We may take $(a, b, c, d, e, f) = (6, 3, 4, 6, 2, 5)$. The cubic surface $\mathcal{F}_{6,3,4,6}$ is given by the equation*

$$X_0^2X_2 + 3X_1^2X_2 + 3X_1^2X_3 + X_0X_2^2 + 2X_1X_2^2 + X_1X_3^2$$
$$+8X_0X_1X_2 + 8X_0X_1X_3 + 5X_0X_2X_3 = 0$$

It has 9 Eckardt points. We find that $(p_0, p_1, p_2, p_3) = (5, 6, 4, 3) \equiv (6, 2, 5, 1)$. The equation of the quartic curve is

$$5x_1^4 + 6x_1^3x_2 + 3x_1^3x_3 + 8x_1x_2^3 + 4x_1x_3^3 + 5x_2^4 + 8x_2^3x_3 + 6x_2x_3^3 + 7x_3^4 = 0.$$

The curve is isomorphic to the Hermitian curve with equation

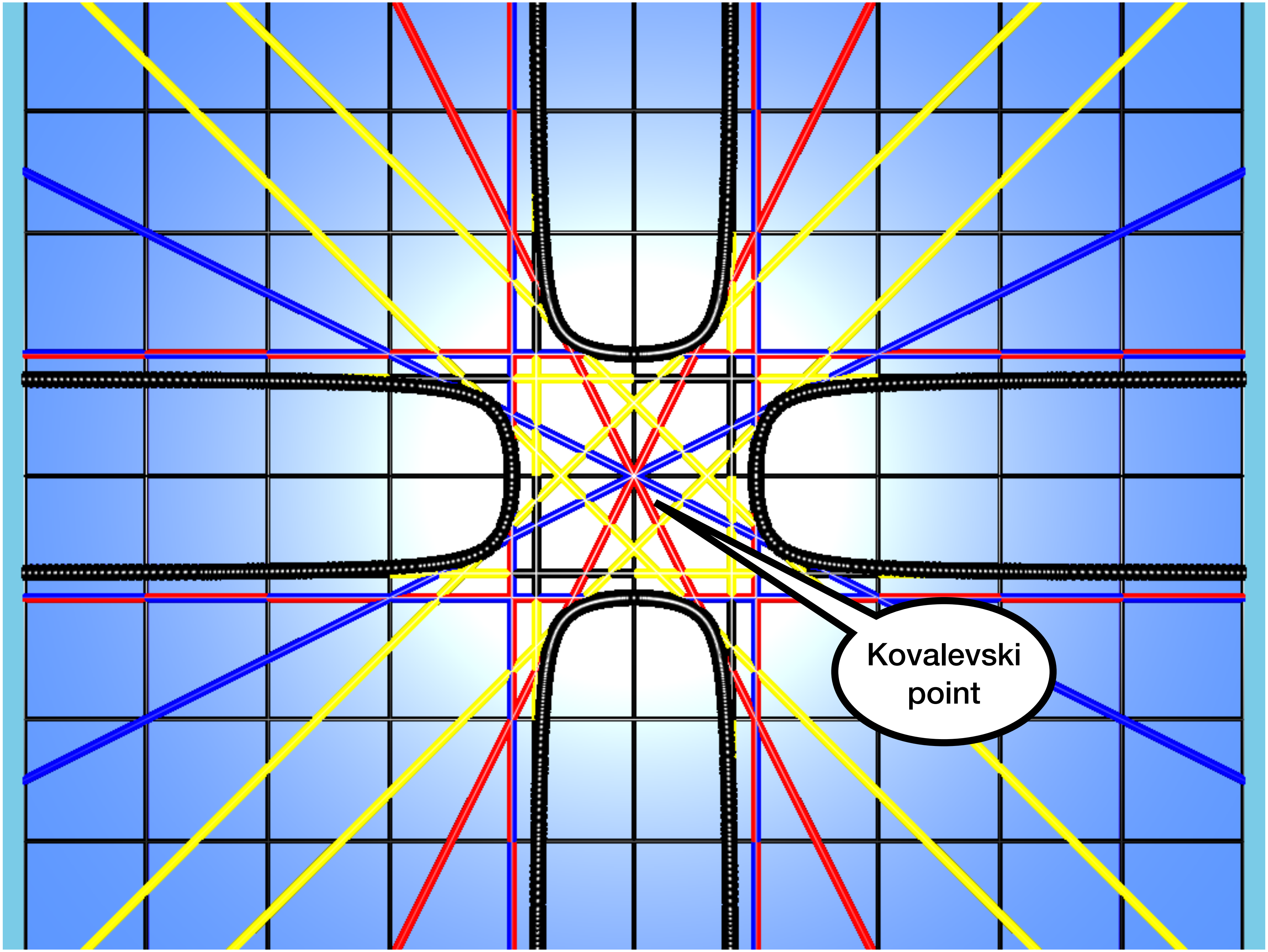
$$x_1^4 + x_2^4 + x_3^4 = 0,$$

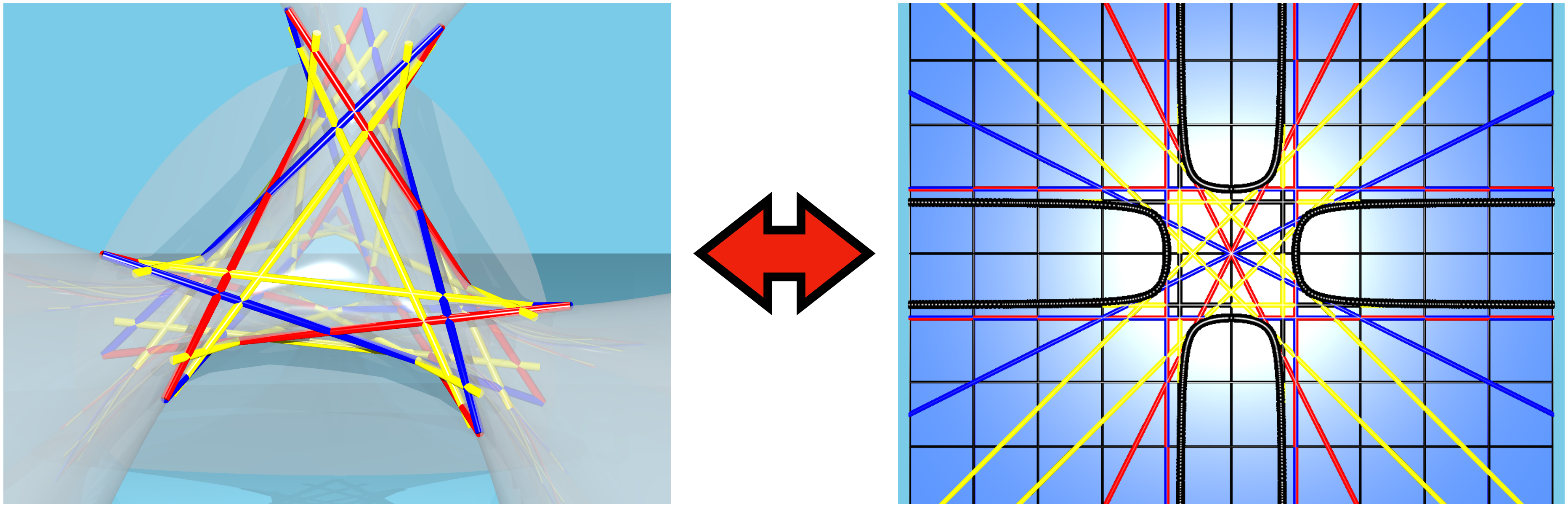
Properties:
Kovalevski Points

Kovalevski Points

- A Kovalevski point is a point (off the quartic) where 4 bitangents meet.

the quartic curve is in black:



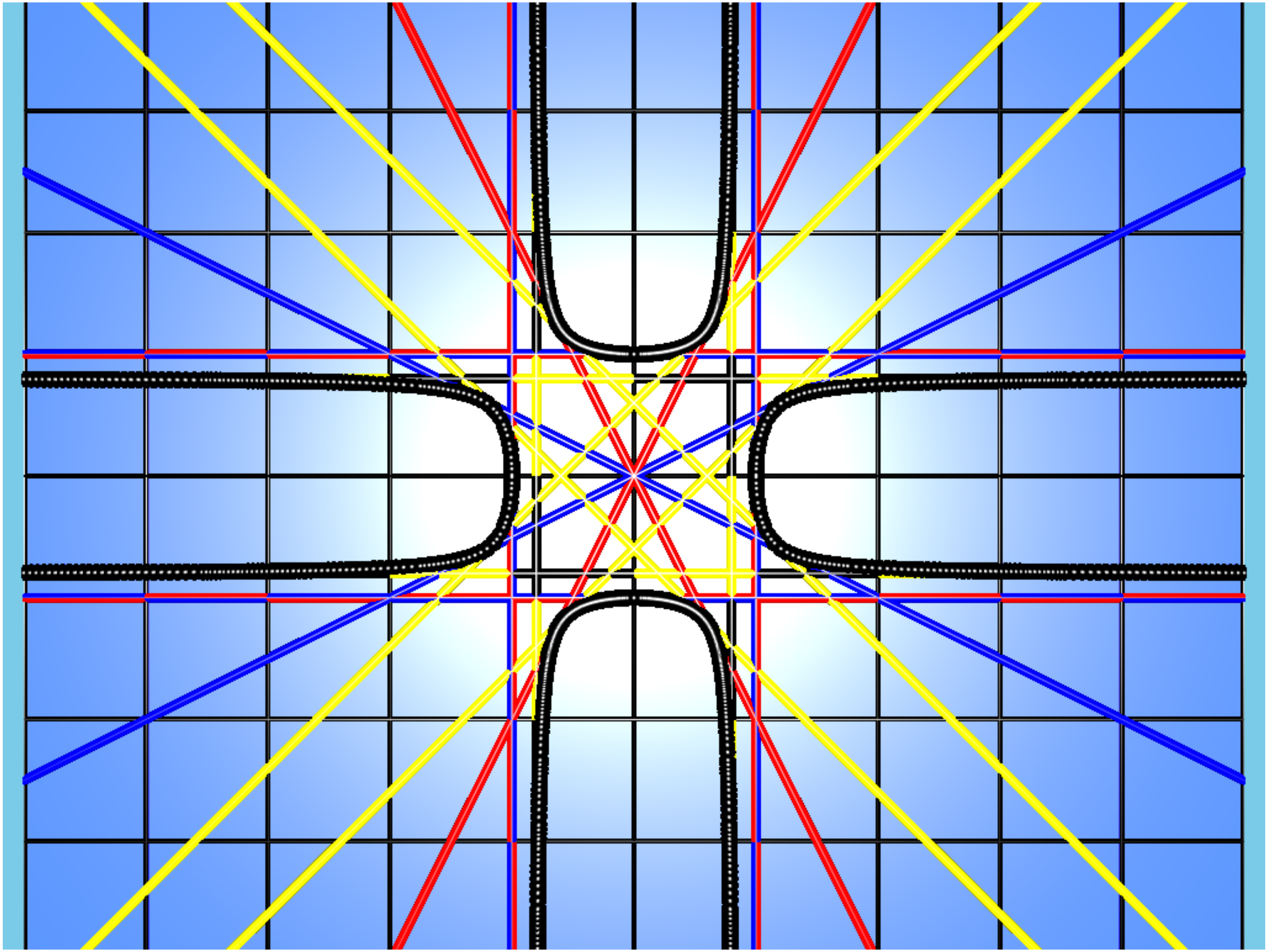


Cubic surface with 27 lines | **quartic curve with 28 bitangents**

The Eckardt surface

(model by Brendan Looi)

the quartic curve is in black:



Kovalevski Points

- The Hermitian curve has 63 Kovalevski points.
- The incidence structure between bitangents and Kovalevski points is the classical until.
- Most quartic curves over finite fields have far fewer Kovalevski points:

Kovalevski Points

The table shows the number of non-isomorphic quartic curves over a given Fq with a given number of Kovalevski points:

q	K63	K21	K15	K9	K7	K5	K3	K1	K0
9	1	0	0	0	0	0	0	0	0
11	0	1	0	0	0	0	0	0	0
13	0	0	0	1	1	0	0	0	0
17	0	0	1	2	0	1	2	1	0
19	0	0	0	3	0	3	2	5	1
23	0	1	0	3	0	6	10	15	6
25	0	1	1	1	1	6	9	16	10
27	0	0	0	3	0	3	6	17	11
29	0	1	0	4	1	10	30	69	60
31	0	0	0	6	0	15	33	108	108
37	0	1	0	5	2	22	72	281	462
41	0	0	1	8	1	28	107	477	1015
43	0	1	0	7	0	37	121	614	1454
47	0	0	0	11	0	44	170	952	2792
49	0	0	1	6	2	28	110	608	1946

Kovalevski vs Eckardt

- Let K = # Kovalevski points of a quartic curve
- Let E = # Eckardt points of a cubic surface
- Question:
- How are K and E related?
- Example: for the Hermitian curve, we have E=9 and K=63.

The Hermitian Curve

$$x_1^4 + x_2^4 + x_3^4 = 0$$

- K = 63, E = 9
- Associated cubic surface:
- $X_1^3 + \alpha X_0^2 X_3 + X_0 X_2^2 + \alpha^2 X_1 X_3^2 = 0$ (E=9).
- quartic curve equation (isomorphic to the Hermitian curve):
  $x_2^4 - \alpha x_1^3 x_3 + (\alpha + 1) x_1 x_3^3 = 0$ (K=63).
- The relation between cubic surfaces ad quartic curves is **many-to-many.**
- Only in this case is there a **one-one** match.

The Kovalevski configuration

$$x_1^4 + x_2^4 + x_3^4 = 0$$

- The associated surface has a unique point P not on any line.
- two cases: 4 = 3 + 1 or 4 = 2+2.
- 9 Kovalevski points arise from the Eckardt points. Namely, they all lie on the tangent plane of P. The tangent plane becomes a bitangent line, so 3+1=4. The 9 Eckardt points form a Hesse configuration AG(2,3).
- The remaining 54 Kovalevski points arise from pairs of double points (points on exactly two lines of the surface). Under projection from P, the pairs fall on top of each other. There are 108 pairs. In this case, 2+2=4.
- 9+54 = 63.

The Kovalevski configuration

$$x_1^4 + x_2^4 + x_3^4 = 0$$

- The design is flag-transitive.
- This means it can be described as coset geometry inside the unitary group.
- What about K=21 and K=15 (and the other values)?
- More work needs to be done!

Relation E vs. K

The following relations between E and K have been observed. There might be more:

	K63	K21	K15	K9	K7	K5	K3	K1	K0
E18	0	0	0	0	0	0	1	1	1
E10	0	0	0	0	0	0	1	1	1
E9	1	0	0	1	1	0	1	1	1
E6	0	1	1	1	0	1	1	1	1
E4	0	0	1	1	1	1	1	1	1
E3	0	1	0	1	1	1	1	1	1
E2	0	0	0	1	1	1	1	1	1
E1	0	0	1	1	1	1	1	1	1
E0	0	0	0	0	0	1	1	1	1

More Information

- We can now give more information. For each curve, we display only one related cubic surface (there may be multiple).

q	Iso	abcdef	E	K	pts	surf	pt-orb	$G_{\mathcal{F}}$	ι	$G_{\mathcal{C}}$
9	0	6,3,4,6,2,5	9	63	28	1	0	432	1	12096
11	0	8,5,3,2,10,7	6	21	0	0	0	24	4	168
13	0	10,3,7,4,11,12	4	9	8	0	0	12	2	24
13	1	10,3,7,4,3,9	4	7	4	0	1	12	6	48
17	0	2,7,16,3,15,8	6	3	20	0	0	24	12	4
17	1	2,7,16,3,14,13	6	1	12	0	1	24	24	2
17	2	2,7,16,3,9,4	6	15	12	0	2	24	4	96
17	3	10,16,4,5,6,12	4	9	24	1	2	12	6	24
17	4	10,16,4,5,14,6	4	3	12	1	3	12	12	6
17	5	10,16,4,5,8,2	4	5	8	1	4	12	6	8
17	6	3,11,2,7,8,12	3	9	24	3	11	6	3	24
19	0	2,14,15,18,13,6	2	1	16	0	0	4	4	2
19	1	2,14,15,18,7,8	2	3	8	0	1	4	4	6
19	2	2,14,15,18,16,5	2	5	24	0	2	4	2	8
19	3	2,14,15,18,17,11	2	1	20	0	3	4	4	2
19	4	2,14,15,18,13,16	2	1	16	0	5	4	4	2
19	5	2,14,15,18,14,17	2	0	20	0	7	4	4	9
19	6	2,14,15,18,7,10	2	5	24	0	8	4	2	8
19	7	2,14,15,18,4,12	2	3	16	0	10	4	2	4
19	8	2,14,15,18,8,9	2	5	16	0	11	4	4	8
19	9	2,14,15,18,18,11	2	1	28	0	12	4	2	6
19	10	2,14,15,18,11,7	2	1	12	0	15	4	4	2
19	11	2,14,15,18,9,15	2	9	8	0	23	4	2	24
19	12	2,14,15,18,6,7	2	9	32	0	24	4	2	24
19	13	11,13,12,15,13,7	4	9	8	1	6	12	2	24

q	Iso	abcdef	E	K	pts	surf	pt-orb	$G_{\mathcal{F}}$	ι	$G_{\mathcal{C}}$
23	5	2,9,8,19,3,11	2	1	16	0	6	4	4	2
23	6	2,9,8,19,7,22	2	3	12	0	7	4	4	6
23	7	2,9,8,19,17,15	2	1	24	0	8	4	4	2
23	8	2,9,8,19,10,13	2	0	20	0	9	4	4	1
23	9	2,9,8,19,19,11	2	3	32	0	10	4	4	4
23	10	2,9,8,19,15,16	2	0	28	0	11	4	4	1
23	11	2,9,8,19,20,4	2	0	16	0	13	4	4	1
23	12	2,9,8,19,13,7	2	3	36	0	15	4	2	6
23	13	2,9,8,19,6,10	2	1	20	0	16	4	4	2
23	14	2,9,8,19,11,13	2	1	16	0	18	4	4	2
23	15	2,9,8,19,20,18	2	1	16	0	21	4	4	2
23	16	2,9,8,19,5,2	2	0	20	0	24	4	4	1
23	17	2,9,8,19,17,20	2	0	24	0	27	4	4	1
23	18	2,9,8,19,11,3	2	1	32	0	30	4	4	2
23	19	2,9,8,19,10,16	2	1	16	0	33	4	4	2
23	20	2,9,8,19,14,2	2	1	20	0	37	4	4	2

q	Iso	abcdef	E	K	pts	surf	pt-orb	$G_{\mathcal{F}}$	ι	$G_{\mathcal{C}}$
23	21	2,9,8,19,18,6	2	1	24	0	38	4	2	2
23	22	2,9,8,19,11,16	2	1	24	0	39	4	4	2
23	23	2,9,8,19,17,16	2	9	24	0	42	4	2	24
23	24	2,9,8,19,20,12	2	3	16	0	45	4	2	4
23	25	2,9,8,19,20,14	2	5	16	0	46	4	2	8
23	26	2,9,8,19,14,3	2	5	16	0	47	4	2	8
23	27	13,4,19,18,15,11	4	1	28	1	2	12	12	2
23	28	13,4,19,18,11,16	4	1	20	1	7	12	6	2
23	29	13,4,19,18,5,3	4	3	24	1	12	12	12	4
23	30	14,7,18,14,10,15	6	9	24	2	4	24	4	24
23	31	14,7,18,14,11,12	6	5	16	2	5	24	12	8
23	32	3,21,8,12,10,7	1	3	16	3	9	2	2	4
23	33	3,21,8,12,4,11	1	9	24	3	24	2	1	24
23	34	3,21,8,12,5,13	1	1	24	3	29	2	1	2
23	35	3,21,8,12,9,15	1	5	40	3	66	2	2	8
23	36	3,21,8,12,15,19	1	3	24	3	74	2	2	6
23	37	3,21,8,12,10,4	1	3	12	3	80	2	2	6
23	38	3,21,8,12,9,8	1	5	24	3	82	2	2	8
23	39	17,10,9,11,11,8	1	5	40	5	17	2	2	8
23	40	4,20,3,4,9,14	3	21	0	13	30	6	1	168

Comments

- Curves 11#0 and 23#40 cause difficulty for the classification algorithm because they don't have any rational points.

- The stabilizer of the surface and the stabilizer of the quartic curve share a common subgroup.

- Namely:

  the stabilizer of the point P inside the group of the surface

  =

  the stabilizer of the distinguished bitangent arising from the tangent plane at P.

Thank you for your attention.

THE END