Some constructions of asymptotically optimal cyclic subspace codes

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Subspace Codes

R. Koetter, F.R. Kschischang. Coding for errors and erasures in random network coding, *IEEE Transaction on Information Theory*, **2008**.

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$S, T \in \mathcal{G}_q(n,k)$ $\quad d(S,T) = 2k - 2\dim_{\mathbb{F}_q}(S \cap T)$ ***Subspace distance***

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parameters of $\mathscr{C}$ $\quad [n, d(\mathscr{C}), |\mathscr{C}|, k]_q$

Cyclic subspace codes

$\mathscr{C} \subseteq \mathscr{G}_q(n,k)$ constant dimension subspace code in $\mathbb{F}_{q^n}$

- **T. Etzion and A. Vardy.** Error-correcting codes in projective space, *IEEE Transaction on Information Theory,* **2011**.

Cyclic subspace codes

$\mathscr{C} \subseteq \mathscr{G}_q(n,k)$ constant dimension subspace code in $\mathbb{F}_{q^n}$

$\mathscr{C}$ is cyclic if

$$\mathscr{C} = \cup_V \, \mathrm{Orb}(V) = \cup_V \, \{\alpha V \,:\, \alpha \in \mathbb{F}_{q^n}^*\}$$

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Cyclic subspace codes

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cyclic subspace code

- **Minimum distance**

$$2 \leq d(\mathscr{C}) \leq 2k$$

- $d = 2k$

Cyclic subspace codes

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cyclic subspace code

- **Minimum distance**

$$2 \leq d(\mathscr{C}) \leq 2k$$

- $d = 2k \Rightarrow k \mid n$

$$\mathscr{C} = \operatorname{Orb}(\mathbb{F}_{q^k})$$

$$|\mathscr{C}| = \frac{q^n - 1}{q^k - 1}$$

F. Manganiello, E. Gorla & J. Rosenthal. Spread codes and spread decoding in network coding. In *2008 IEEE International Symposium on Information Theory* (pp. 881-885). IEEE, **2008**.

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$$\mathscr{C} = \operatorname{Orb}(\mathbb{F}_{q^k})$$

$$|\mathscr{C}| = \frac{q^n - 1}{q^k - 1}$$

* $d = 2k - 2$

$$\mathscr{C} = \operatorname{Orb}(S) \Longrightarrow |\mathscr{C}| = \frac{q^n - 1}{q - 1}$$

Cyclic subspace codes

$$\mathscr{C} = \cup_V \operatorname{Orb}(V) = \cup_V \{\alpha V \,:\, \alpha \in \mathbb{F}_{q^n}^*\}$$

cyclic subspace code

* **Cardinality**

$\mathscr{A}_q(n, d, k) :=$ **max # of codewords a subspace code** $\mathscr{C} \subseteq \mathscr{G}_q(n, k)$ **may have**

Bounds on the size of constant-dimension subspace codes

$\mathscr{A}_q(n,d,k) :=$ **max # of codewords a subspace code $\mathscr{C} \subseteq \mathscr{G}_q(n,k)$ may have**

Sphere-packing bound

$$\mathscr{A}_q(n,d,k) \leq \left\lfloor \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \left(\sum_{i=0}^{\lfloor (d/2-1)/2 \rfloor} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q \cdot q^{i^2} \right) \right\rfloor$$

Singleton bound

$$\mathscr{A}_q(n,d,k) \leq \begin{bmatrix} n-d/2+1 \\ \max\{k, n-k\} \end{bmatrix}_q$$

R. Koetter, F.R. Kschischang. Coding for errors and erasures in random network coding, *IEEE Transaction on Information Theory*, **2008**.

Anticode bound

$$\mathscr{A}_q(n,d,k) \leq \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} \max\{n-k, k\} + d/2 - 1 \\ d/2 - 1 \end{bmatrix}_q}$$

R. Ahlswede, H. K. Aydinian, L. H. Khachatrian. On perfect codes and related concepts. *Designs, Codes and Cryptography*, *22*(3), 221-237, **2001**.

H. Wang, C. Xing, R. Safavi-Naini. Linear authentication codes: bounds and constructions. *IEEE Transactions on Information Theory*, *49*(4), 866-872, **2003**.

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R. Koetter, F.R. Kschischang. Coding for errors and era *Theory*, **2008**.

Singleton bound

S. Kurz. Constructions and bounds for subspace codes. *arXiv preprint arXiv:2112.11766*, **2021**.

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Johnson type bound II

$$\mathscr{A}_q(n,d,k) \le \left\lfloor \frac{q^n-1}{q^k-1} \left\lfloor \frac{q^{n-1}-1}{q^{k-1}-1} \left\lfloor \cdots \left\lfloor \frac{q^{n-k+d/2+1}-1}{q^{d/2+1}-1} \left\lfloor \frac{q^{n-k+d/2}-1}{q^{d/2}-1} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor$$

T. Xia and F.-W. Fu. Johnson type bounds on constant dimension codes. *Designs, Codes and Cryptography,* 50:163–172, **2009**.

Khaleghi, D. Silva, and F. R. Kschischang. Subspace codes. In *IMA International Conference on Cryptography and Coding*, pages 1–21. Springer, **2009.**

T. Etzion and A. Vardy. Error-correcting codes in projective space, *IEEE Transaction on Information Theory,* **2011**.

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cyclic subspace code

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$\mathscr{A}_q(n, d, k) :=$ **max # of codewords a subspace code $\mathscr{C} \subseteq \mathscr{G}_q(n, k)$ may have**

- **When $d = 2k - 2$**

Johnson type bound II

$$\mathscr{A}_q(n, 2k-2, k) \leq \left\lfloor \frac{q^n - 1}{q^k - 1} \left\lfloor \frac{q^{n-1} - 1}{q^{k-1} - 1} \right\rfloor \right\rfloor := J_q(n, 2k-2, k)$$

Largest known constructions of cyclic subspace codes in $\mathcal{G}_q(rk, k)$ with minimum distance 2k-2

Johnson type bound II

$$\mathcal{A}_q(rk,2k-2,k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(rk,2k-2,k) \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

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as k (or q) $\to +\infty$

$\mathcal{F}$ **family of codes of sizes** $S(rk,k,q)$ **is Asymptotically optimal if**

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(rk,k,q)}{J_q(rk,2k-2,k)} = 1$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2}q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2}q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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R. M. Roth, N. Raviv, & I. Tamo. Construction of Sidon spaces with applications to coding. *IEEE Transactions on Information Theory*, 64(6), 4412-4422, **2018**.

Theorem (Roth, Raviv, Tamo, 2018)

$n = 2k, q \geq 3, \omega$ primitive element of $\mathbb{F}_{q^k}$

$\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^k}$ s.t. $\mathrm{N}_{q^{2k}/q^k}(\gamma) = \omega$, for $h = 1, \ldots, \left\lfloor \frac{q-1}{2} \right\rfloor$.

Let

$$_2V_{k,h} := \{v + v^q \omega^h \gamma : v \in \mathbb{F}_{q^k}\} \subseteq \mathbb{F}_{q^{2k}} = \mathbb{F}_{q^n}.$$

The code

$$_2\mathcal{C}_k = \bigcup_{h=1}^{\tau} \mathrm{Orb}(_2V_{k,h}) \subseteq \mathcal{G}_q(2k, k)$$

has minimum distance $2k - 2$ and size $\left\lfloor \frac{q-1}{2} \right\rfloor \frac{q^{2k}-1}{q-1}$.

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$n=4k$	$S_2(4k,k,q)=\lfloor \frac{q^k-2}{2}\rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2}q^{6k}$
$n=rk$, $r\geq 3$	$S_3(rk,k,q)=q^k\frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell=(\lceil \frac{r}{2}\rceil -2)$	$\sim q^{(\lceil \frac{r-1}{2}\rceil+r)k}$
$n=rk$, $r=2h+1$, $h\geq 2$	$S_4(rk,k,q)=h\left((q^k-1)^h(q^{rk}-1)+\frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2}\rfloor+r)k}$
$n=rk$, $r=2h+1$, $h\geq 2$	$S_5(rk,k,q)=hq^k(q^k-1)^{h-1}(q^{rk}-1)+\frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2}\rfloor+r)k}$

Johnson type bound II

$$\mathcal{A}_q(2k,2k-2,k)\leq \left\lfloor \frac{q^{2k}-1}{q^k-1}\left\lfloor \frac{q^{2k-1}-1}{q^{k-1}-1}\right\rfloor\right\rfloor = J_q(2k,2k-2,k)\sim q^{2k}$$

as k (or q) $\to +\infty$

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(2k,k,q)}{J_q(2k,2k-2,k)}=\frac{1}{2}$$

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Theorem (Roth, Raviv, Tamo, 2018)

$n=2k, q\geq 3, \omega$ primitive element of $\mathbb{F}_{q^k}$

$\gamma\in\mathbb{F}_{q^n}\backslash\mathbb{F}_{q^k}$ s.t. $\mathrm{N}_{q^{2k}/q^k}(\gamma)=\omega$, for $h=1,\ldots,\left\lfloor \frac{q-1}{2}\right\rfloor$.

Let

$${}_2V_{k,h}:=\{v+v^q\omega^h\gamma:\ v\in\mathbb{F}_{q^k}\}\subseteq \mathbb{F}_{q^{2k}}=\mathbb{F}_{q^n}\,.$$

The code

$${}_2\mathscr{C}_k=\bigcup_{h=1}^{\tau}\mathrm{Orb}({}_2V_{k,h})\subseteq\mathcal{G}_q(2k,k)$$

has minimum distance $2k-2$ and size $\left\lfloor \frac{q-1}{2}\right\rfloor \frac{q^{2k}-1}{q-1}$.

Largest known constructions of cyclic subspace codes in $\mathcal{G}_q(rk, k)$ with minimum distance 2k-2

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2}q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2}q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{\ell k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

Johnson type bound II

$$\mathcal{A}_q(4k, 2k-2, k) \leq \left\lfloor \frac{q^{4k}-1}{q^k-1} \left\lfloor \frac{q^{4k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(4k, 2k-2, k) \sim q^{6k}$$

as k (or q) $\to +\infty$

S. Yu, L. Ji. Two new constructions of cyclic subspace codes via Sidon spaces. *Designs, Codes and Cryptography*, *92*(11), 3799-3811, **2024**.

$$\lim_{k (\text{or } q) \to +\infty} \frac{S(4k, k, q)}{J_q(4k, 2k-2, k)} = \frac{1}{2}$$

Largest known constructions of cyclic subspace codes in $\mathcal{G}_q(rk, k)$ with minimum distance 2k-2

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
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$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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$$\mathcal{A}_q(rk, 2k-2, k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(rk, 2k-2, k) \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k (\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk,2k-2,k)} = 1 \Leftrightarrow r = 3$$

Zhang, H., Tang, C., & Cao, X. Large optimal cyclic subspace codes. *Discrete Mathematics*, *347*(7), 114007, **2024**.

Theorem (Zhang, Tang, Cao, 2024)

$n = rk, r \geq 3,\ \ell := \left\lceil \frac{r}{2} \right\rceil - 2,\ \omega$ primitive element of $\mathbb{F}_{q^k}$

$\gamma \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^k}$ s.t. $\mathbb{F}_{q^{2k}} = \mathbb{F}_{q^k}(\gamma)$

$I_{k,\ell_0,\ell} := \{(\alpha_1, \ldots, \alpha_{\ell_0}) \in \mathbb{F}_{q^k}^{\ell_0} :\ \alpha_1, \ldots, \alpha_{\ell_0 - 1} \in \mathbb{F}_{q^k}, \alpha_{\ell_0} \in \mathbb{F}_{q^k}^*\} \times \mathbb{F}_{q^k} \times \{0,1,\ldots,q-2\} \times \{\ell_0, \ell_0+1, \ldots, \ell\}$

For $0 \leq \ell_0 \leq \ell$, define

$${}_rV_{k,\underline{h}} = \{uP_{k,\underline{\alpha_{\ell_0}}}(\gamma) + \omega^j(u^q + au)\gamma^{\ell_1+1} : u \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(rk,k) \text{ for any}$$

$$\underline{h} = (\underline{\alpha_{\ell_0}}, a, j, \ell_1) \in I_{k,\ell_0,\ell}.$$

The code

$${}_r\mathcal{C}_k = \bigcup_{\ell_0=0}^{\ell} \bigcup_{\underline{h} \in I_{k,\ell_0,\ell}} \mathrm{Orb}({}_rV_{k,\underline{h}})$$

has minimum distance $2k-2$ and size $q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$.

Take away:

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(2k,k,q)}{J_q(2k,2k-2,k)} = \frac{1}{2}$$

$n = 2k$

(Almost) Asymptotically optimal cyclic subspace codes

R. M. Roth, N. Raviv, & I. Tamo. Construction of Sidon spaces with applications to coding. *IEEE Transactions on Information Theory*, 64(6), 4412-4422, **2018**.

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(3k,k,q)}{J_q(3k,2k-2,k)} = 1$$

$n = 3k$

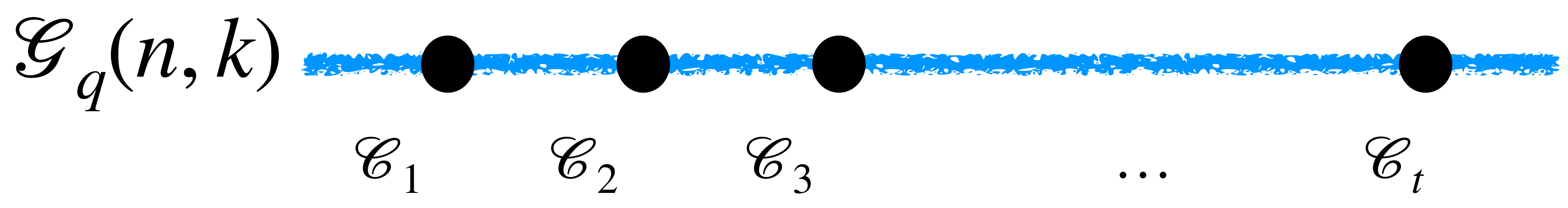
Asymptotically optimal cyclic subspace codes

Zhang, H., Tang, C., & Cao, X. Large optimal cyclic subspace codes. *Discrete Mathematics*, *347*(7), 114007, **2024**.

Largest known constructions of cyclic subspace codes in $\mathcal{G}_q(rk,k)$ with minimum distance 2k-2

$$\mathcal{C} = \cup_i \mathcal{C}_i \subseteq \mathcal{G}_q(n,k)$$

$$\mathcal{C}_i \subseteq \mathcal{G}_q(n,k) \quad \forall i$$



Parameters	Size	Asymptotic behavior
$n=2k$, $q>2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2}q^{2k}$
$n=4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2}q^{6k}$
$n=rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n=rk$, $r=2h+1,\ h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n=rk$, $r=2h+1,\ h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

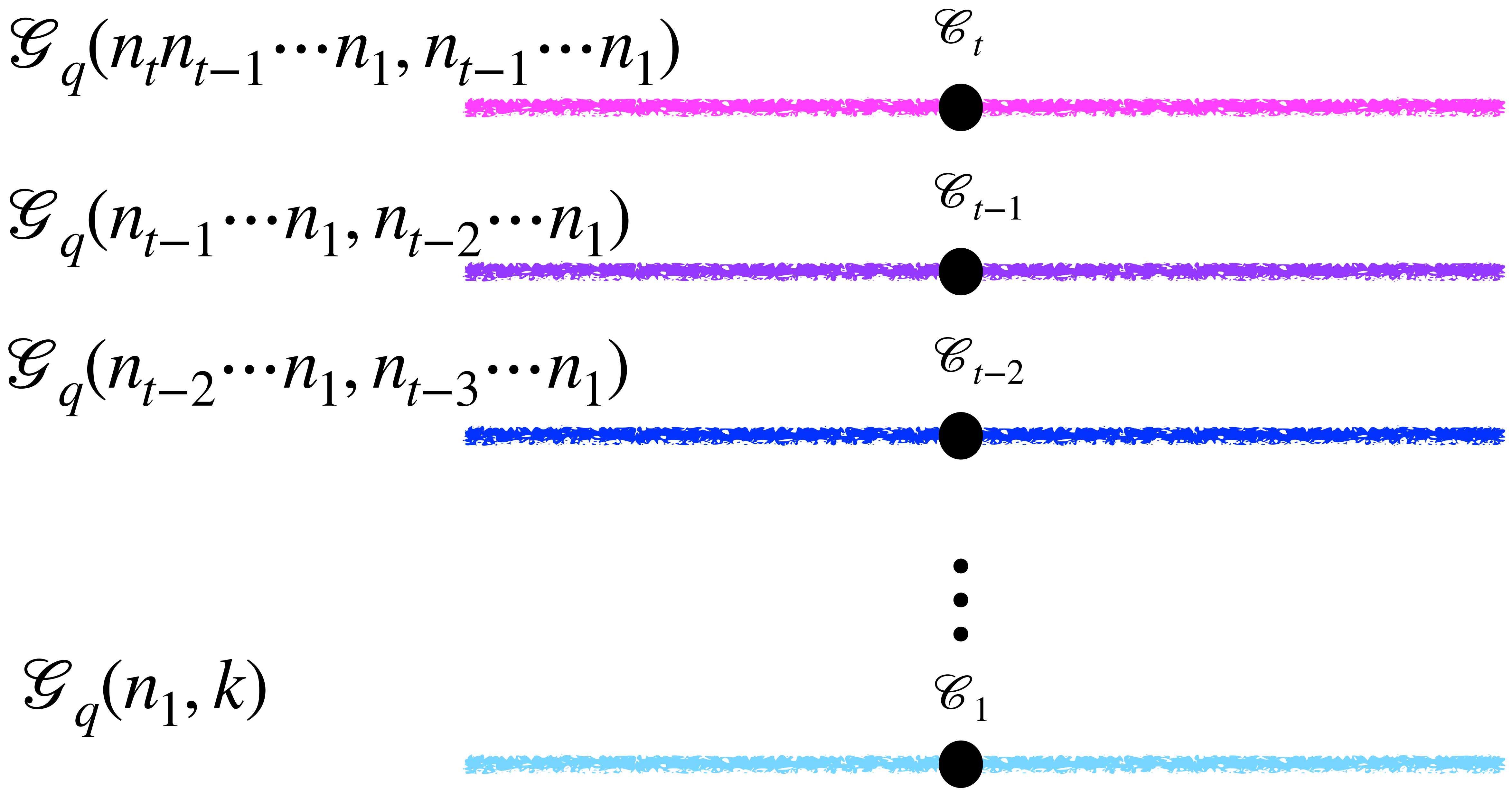
Common approach VS Our approach

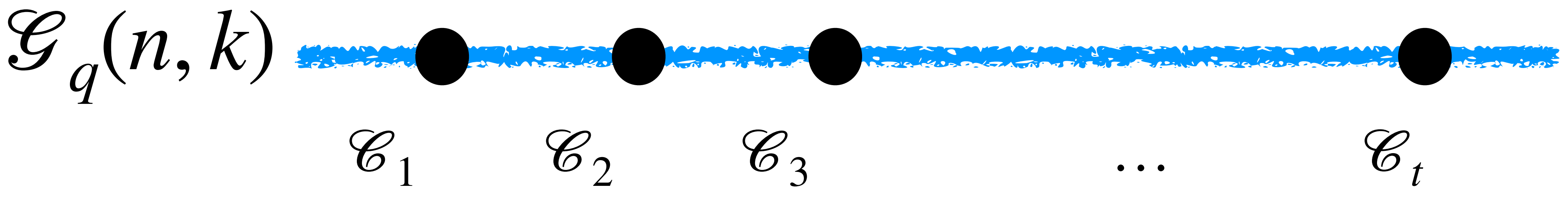
$$\mathscr{C} = \cup_i \mathscr{C}_i \subseteq \mathscr{G}_q(n,k)$$

$$\mathscr{C} = \mathscr{C}_t \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(n,k)$$

$$n = n_t \cdots n_1$$

$$\mathscr{C}_i \subseteq \mathscr{G}_q(n,k) \quad \forall i$$





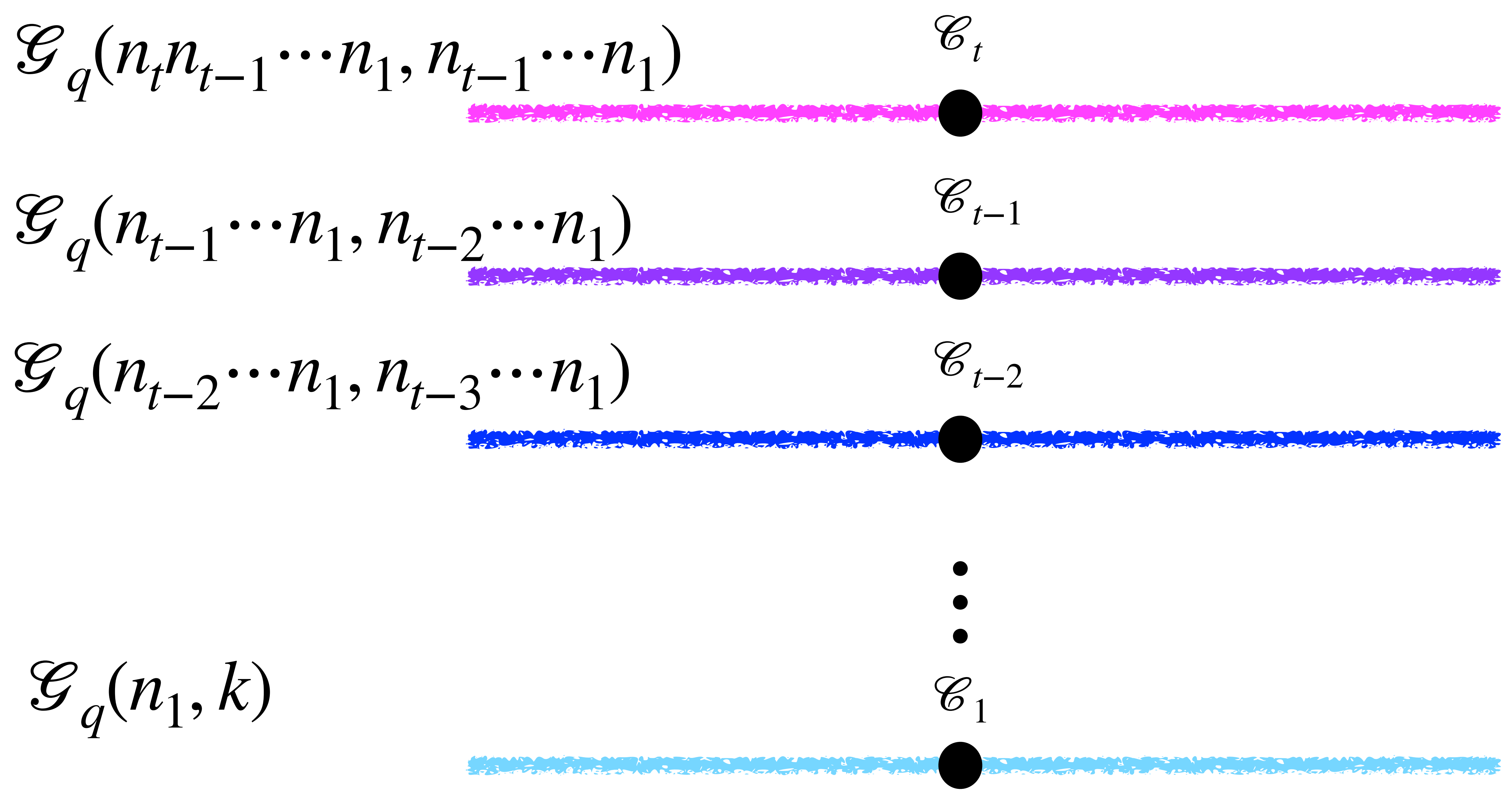
Our goal

$$\mathscr{C} = \mathscr{C}_t \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(n,k)$$
$$n = n_t \cdots n_1$$

New Constructions of **Asymptotically Optimal** cyclic subspace codes $\mathscr{C} \subseteq \mathscr{G}_q(rk,k)$

$$d = 2k - 2$$

$$|\mathscr{C}| \sim q^{2(r-1)k}$$



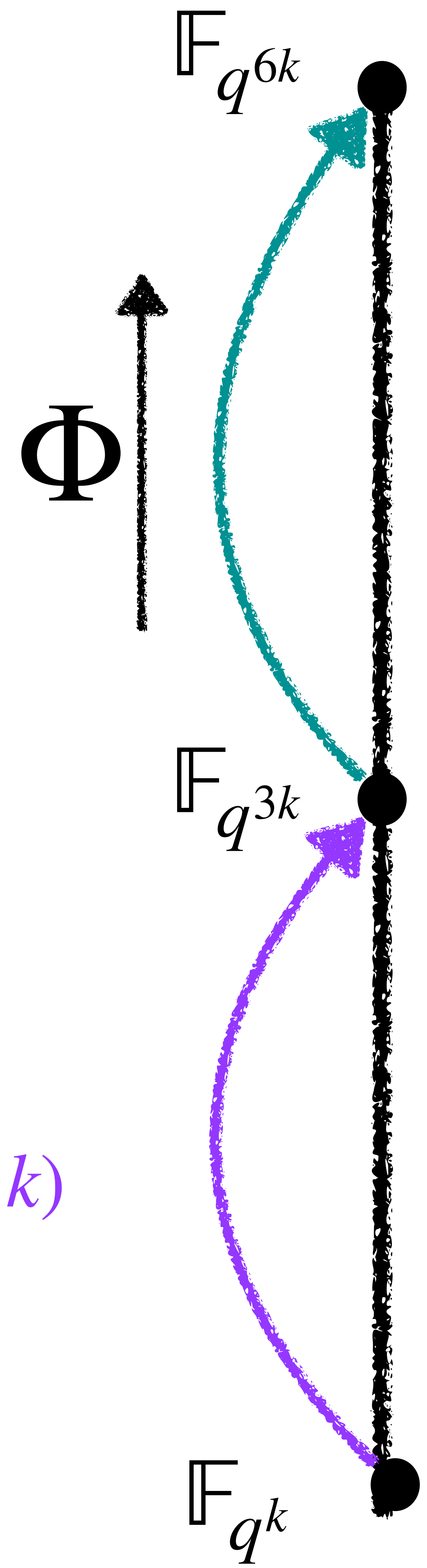
C.C., P. Santonastaso. Asymptotically optimal cyclic subspace codes, https://arxiv.org/abs/2507.09290, **2025**

Nested operation

$$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\eta)$$

$$\mathbb{F}_{q^{6k}} = \mathbb{F}_{q^{3k}}(\gamma)$$

$V_2 = \mathrm{Im}(\Phi) = \{u + u^q\gamma : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(6k, 3k)$



$V_1 = \{v + v^q\eta : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k, k)$

$$\Phi : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{6k}}$$
$$u \mapsto u + u^q\gamma$$

injective

$\mathbb{F}_q$-linear

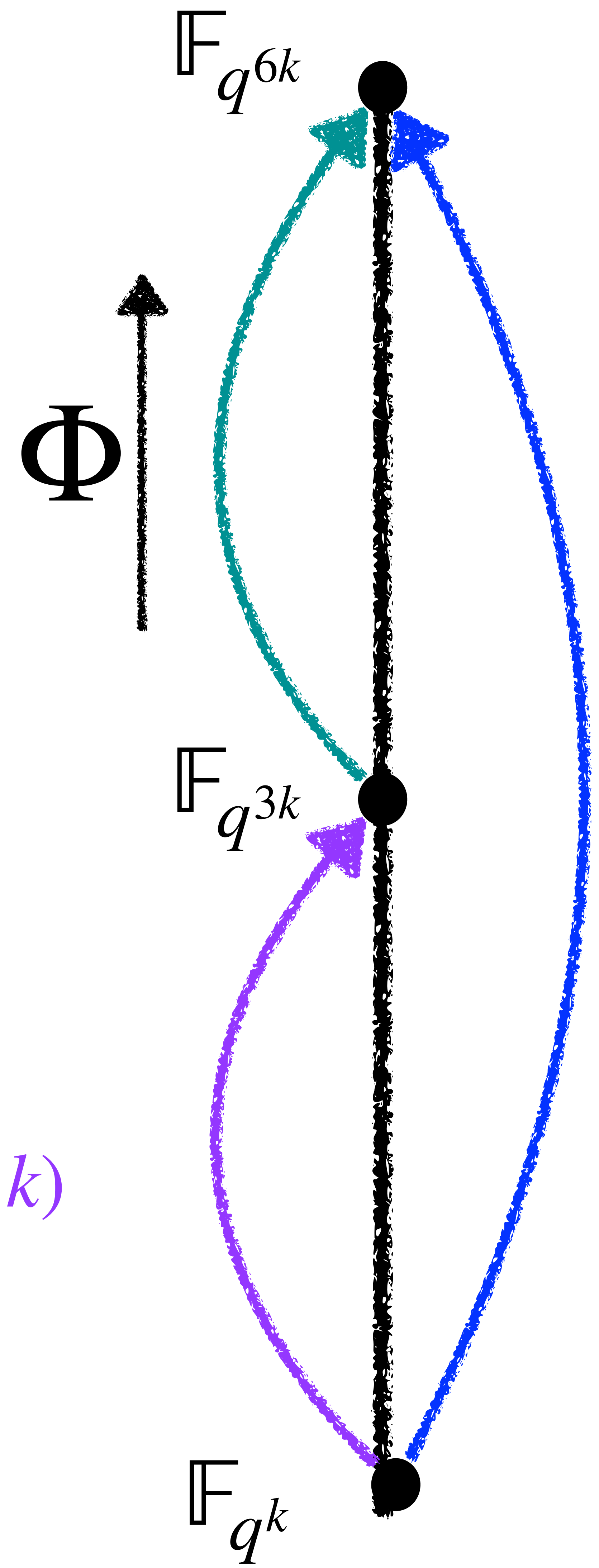
Nested operation

$$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\eta)$$

$$\mathbb{F}_{q^{6k}} = \mathbb{F}_{q^{3k}}(\gamma)$$

$V_2 = \mathrm{Im}(\Phi) = \{u + u^q\gamma : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(6k,3k)$

$V_1 = \{v + v^q\eta : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k,k)$



$$\Phi : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{6k}}$$
$$u \mapsto u + u^q\gamma$$

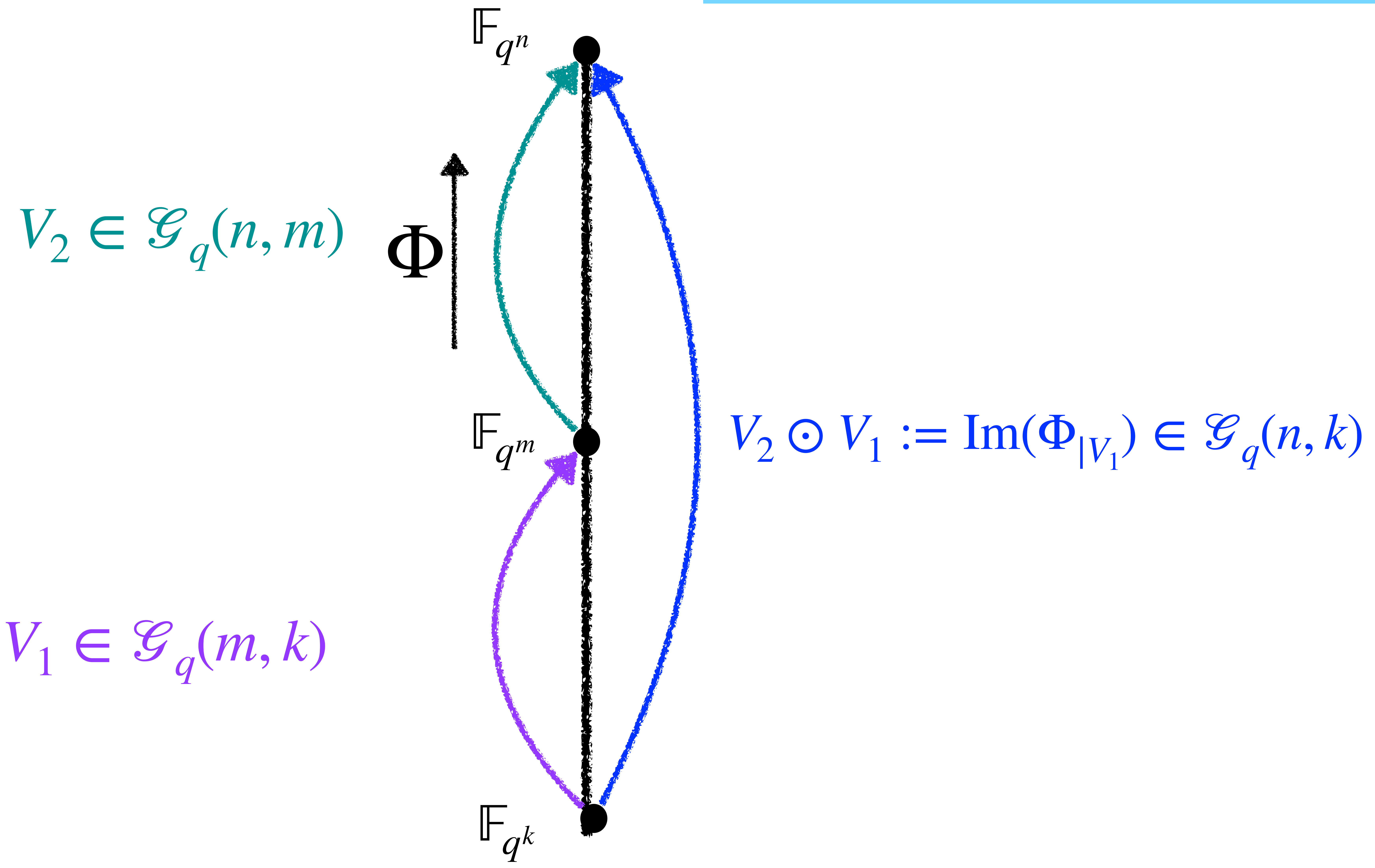
injective

$\mathbb{F}_q$-linear

$$\begin{aligned} V_2 \odot V_1 &:= \{\Phi(u) : u \in V_1\} \\ &= \{u + u^q\gamma : u \in V_1\} \\ &= \{(v + v^q\eta) + (v + v^q\eta)^q\gamma : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(6k,k) \end{aligned}$$

Nested operation

$V_2 = \mathrm{Im}(\Phi)$ $\quad \Phi : \mathbb{F}_{q^m} \to \mathbb{F}_{q^n}$
injective $\mathbb{F}_q$-linear

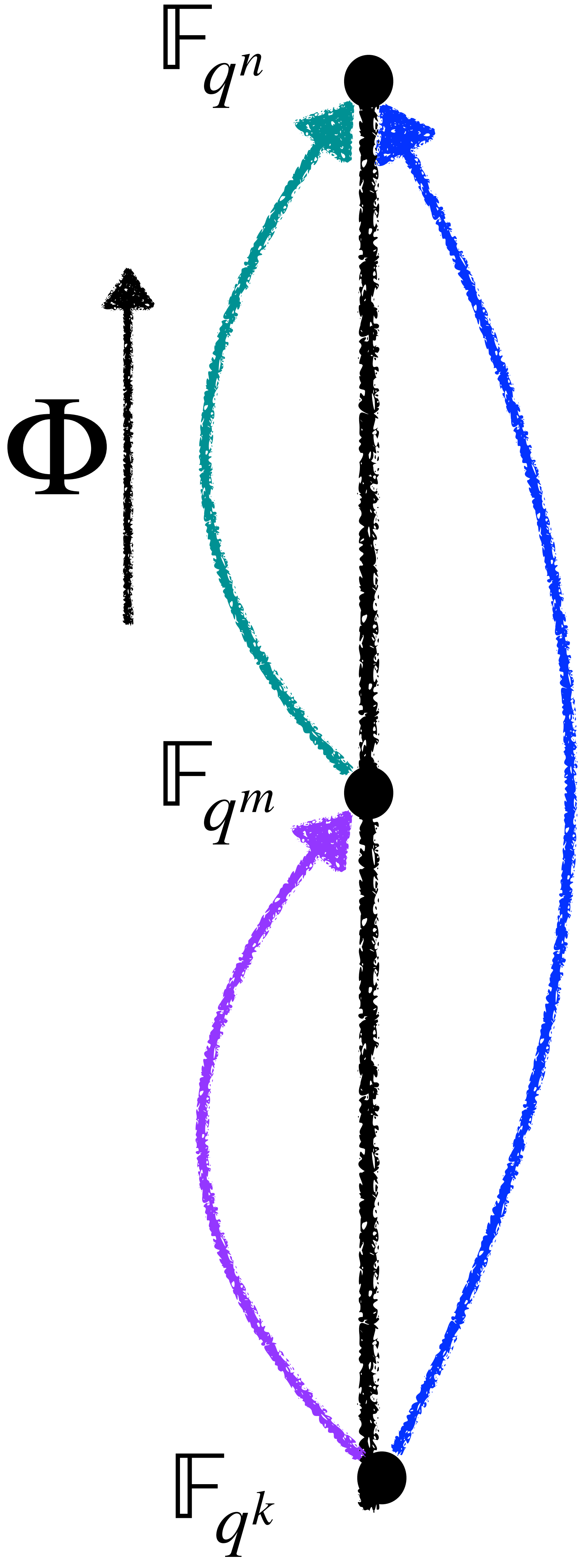


Nested operation

$V_2 = \mathrm{Im}(\Phi)$ $\quad \Phi : \mathbb{F}_{q^m} \to \mathbb{F}_{q^n}$

injective $\mathbb{F}_q$-linear

$V_2 \in \mathcal{G}_q(n, m)$



$V_2 \odot V_1 := \mathrm{Im}(\Phi_{|V_1}) \in \mathcal{G}_q(n, k)$

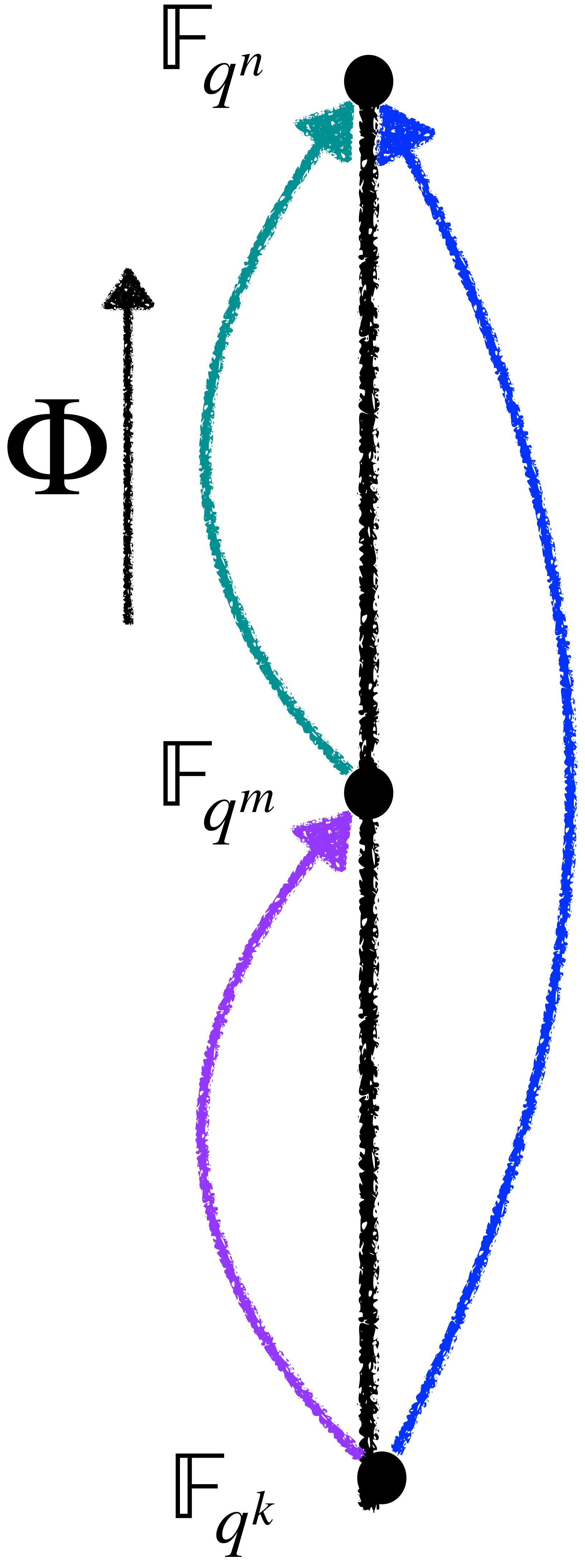
$V_1 \in \mathcal{G}_q(m, k)$

Nested operation

$V_2 = \mathrm{Im}(\Phi)$ $\quad \Phi : \mathbb{F}_{q^m} \to \mathbb{F}_{q^n}$

injective $\mathbb{F}_q$-linear

$V_2 \in \mathcal{G}_q(n, m)$



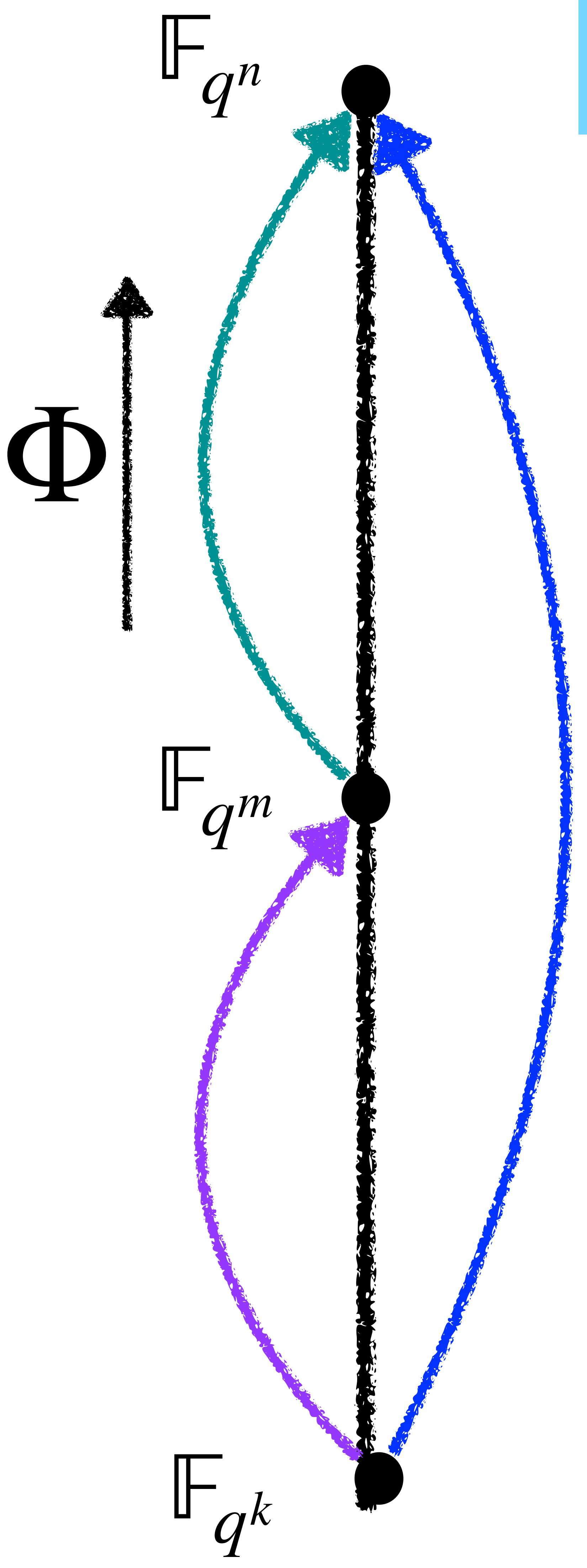
$V_2 \odot V_1 := \mathrm{Im}(\Phi_{|V_1}) \in \mathcal{G}_q(n, k)$

$V_1 \in \mathcal{G}_q(m, k)$

Nested operation

$$V_{2,h_2} = \mathrm{Im}(\Phi_{2,h_2}) \quad \Phi_{2,h_2} : \mathbb{F}_{q^m} \to \mathbb{F}_{q^n}$$
$$\forall h_2 = 1, \ldots, s_2$$

$$\mathscr{C}_2 = \cup_{h_2=1}^{s_2} \mathrm{Orb}(V_{2,h_2}) \subseteq \mathscr{G}_q(n, m)$$



$$\mathscr{C}_2 \odot \mathscr{C}_1 := \bigcup_{h_2=1}^{s_2} \bigcup_{h_1=1}^{s_1} \bigcup_{\alpha \in \mathbb{F}_{q^m}^*} \mathrm{Orb}(V_{2,h_2} \odot \alpha V_{1,h_1}) \subseteq \mathscr{G}_q(n, k).$$

$$\mathscr{C}_1 = \cup_{h_1=1}^{s_1} \mathrm{Orb}(V_{1,h_1}) \subseteq \mathscr{G}_q(m, k)$$

Nested operation

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$

$\mathbb{F}_{q^{9k}} = \mathbb{F}_{q^{3k}}(\gamma_2)$

$V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k, k)$

$V_{2,a_2,j_2} = \mathrm{Im}(\Phi_{2,a_2,j_2}) = \{u + \omega_2^{j_2}(u^q + ua_2)\gamma_2 : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(9k,3k)$

$\omega_1 \in \mathbb{F}_{q^k}$ **Primitive element**

$\omega_2 \in \mathbb{F}_{q^{3k}}$ **Primitive element**

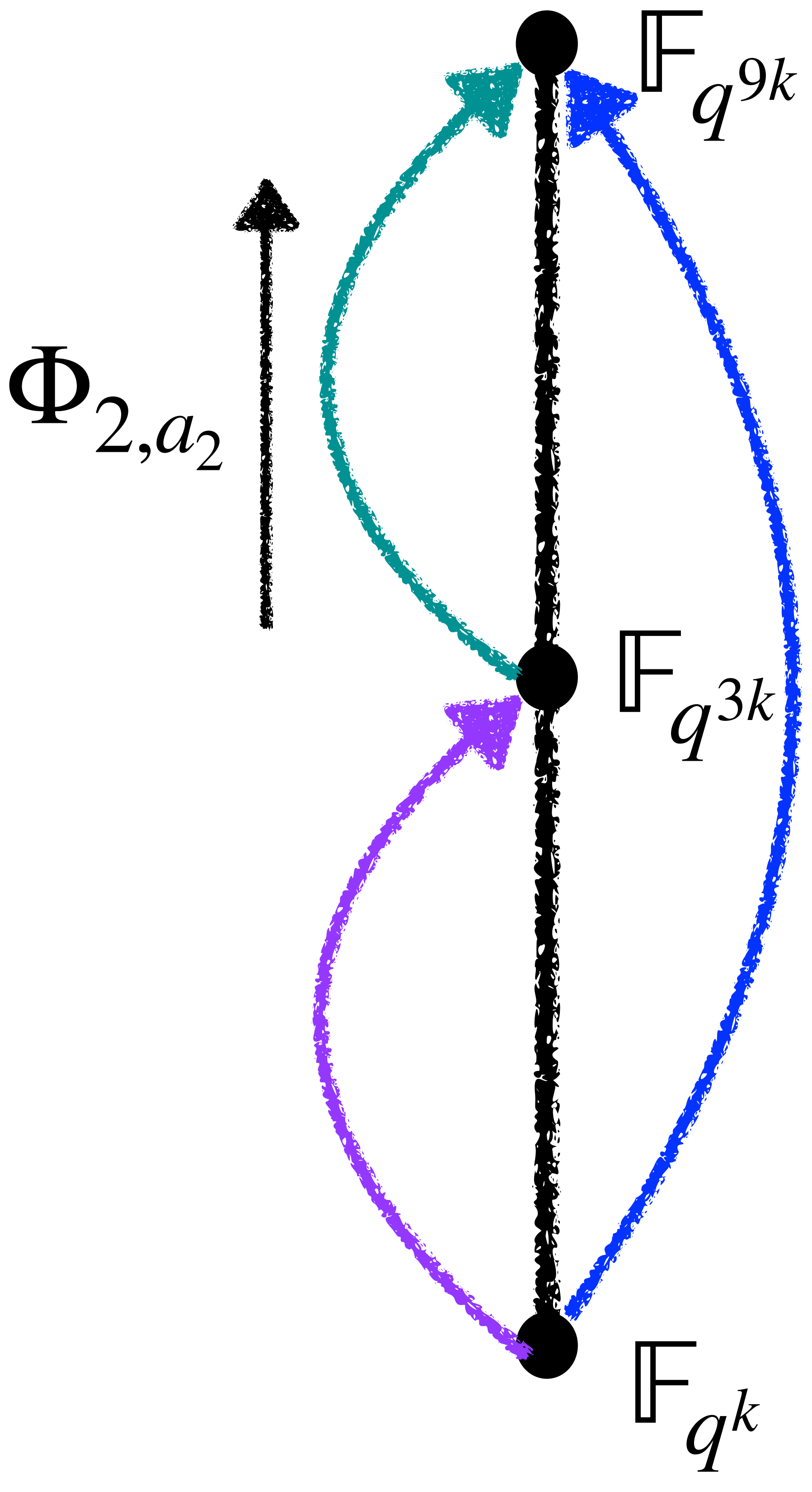
$0 \le j_1, j_2 \le q-2$

$\Phi_{2,a_2,j_2} : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{9k}}$

injective $\quad u \mapsto u + \omega_2^{j_2}(u^q + ua_2)\gamma_2$

$\mathbb{F}_q$**-linear**

$\mathscr{C}_2 = \cup_{0 \le j_1 \le q-2} \cup_{a_2 \in \mathbb{F}_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2}) \subseteq \mathcal{G}_q(9k,3k)$



$\mathscr{C}_2 \odot \mathscr{C}_1 := \bigcup_{0 \le j_1, j_2 \le q-2} \bigcup_{a_2 \in \mathbb{F}_{q^{3k}}} \bigcup_{a_1 \in \mathbb{F}_{q^k}} \bigcup_{\alpha \in \mathbb{F}^*_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2} \odot \alpha V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(9k, k)$

$\mathscr{C}_1 = \cup_{0 \le j_1 \le q-2} \cup_{a_1 \in \mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k, k)$

What is the advantage?

Theorem 1 (C.C., P. Santonastaso)

$\mathscr{C}_1 = \cup_{h_1=1}^{s_1} \mathrm{Orb}(V_{1,h_1}) \subseteq \mathscr{G}_q(m,k),$ $\quad d(\mathscr{C}_1) = 2k-2$

$\mathscr{C}_2 = \cup_{h_2=1}^{s_2} \mathrm{Orb}(V_{2,h_2}) \subseteq \mathscr{G}_q(n,m)$ $\quad d(\mathscr{C}_2) = 2m-2$

$V_{2,h_2} = \mathrm{Im}(\Phi_{2,h_2})$

$\Phi_{2,h_2} : \mathbb{F}_{q^m} \to \mathbb{F}_{q^n}$ **injective** $\mathbb{F}_q$**-linear** $\forall h_2 = 1,\dots,s_2$

$\Downarrow$

$\mathscr{C}_2 \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(n,k)$ $\quad d(\mathscr{C}_2 \odot \mathscr{C}_1) = 2k-2$ $\quad |\mathscr{C}_2 \odot \mathscr{C}_1| = |\mathscr{C}_2||\mathscr{C}_1|$

Nested operation

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$ $\quad V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k, k)$

$\mathbb{F}_{q^{9k}} = \mathbb{F}_{q^{3k}}(\gamma_2)$ $\quad V_{2,a_2,j_2} = \mathrm{Im}(\Phi_{2,a_2,j_2}) = \{u + \omega_2^{j_2}(u^q + ua_2)\gamma_2 : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(9k,3k)$

$\Phi_{2,a_2,j_2} : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{9k}}$

$u \mapsto u + \omega_2^{j_2}(u^q + ua_2)\gamma_2$

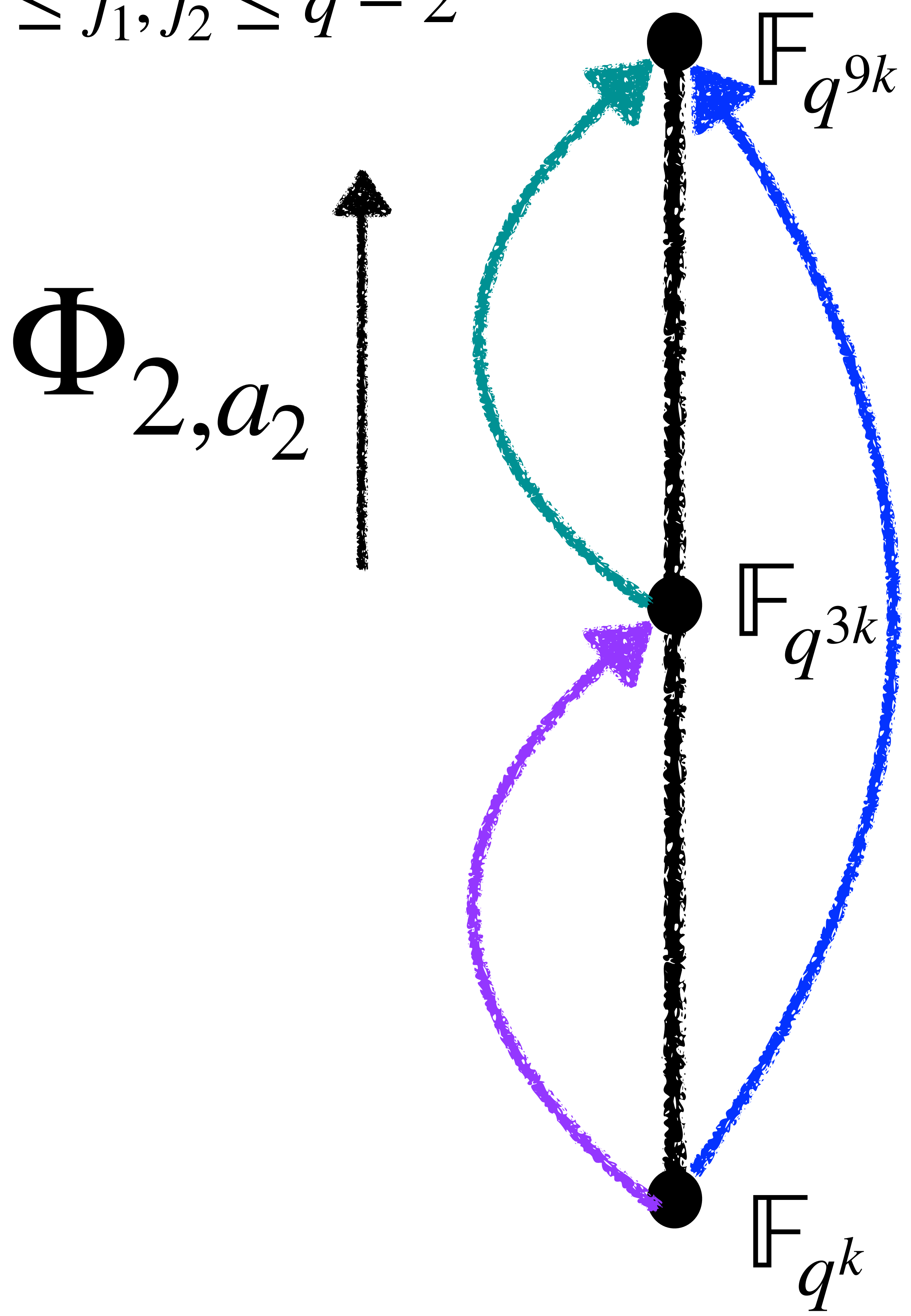
injective

$\mathbb{F}_q$**-linear**

$\omega_1 \in \mathbb{F}_{q^k}$ **Primitive element**

$\omega_2 \in \mathbb{F}_{q^{3k}}$ **Primitive element**

$0 \leq j_1, j_2 \leq q-2$



$\mathcal{C}_1 = \cup_{0\leq j_1\leq q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k, k)$

$\mathcal{C}_2 = \cup_{0\leq j_1\leq q-2} \cup_{a_2\in\mathbb{F}_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2}) \subseteq \mathcal{G}_q(9k,3k)$

$$\mathcal{C}_2 \odot \mathcal{C}_1 := \bigcup_{0\leq j_1,j_2\leq q-2} \bigcup_{a_2\in\mathbb{F}_{q^{3k}}} \bigcup_{a_1\in\mathbb{F}_{q^k}} \bigcup_{\alpha\in\mathbb{F}^*_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2} \odot \alpha V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(9k, k)$$

$$d(\mathcal{C}_2 \odot \mathcal{C}_1) = 2k - 2$$

Nested operation

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$

$\mathbb{F}_{q^{9k}} = \mathbb{F}_{q^{3k}}(\gamma_2)$

$\omega_1 \in \mathbb{F}_{q^k}$ **Primitive element**

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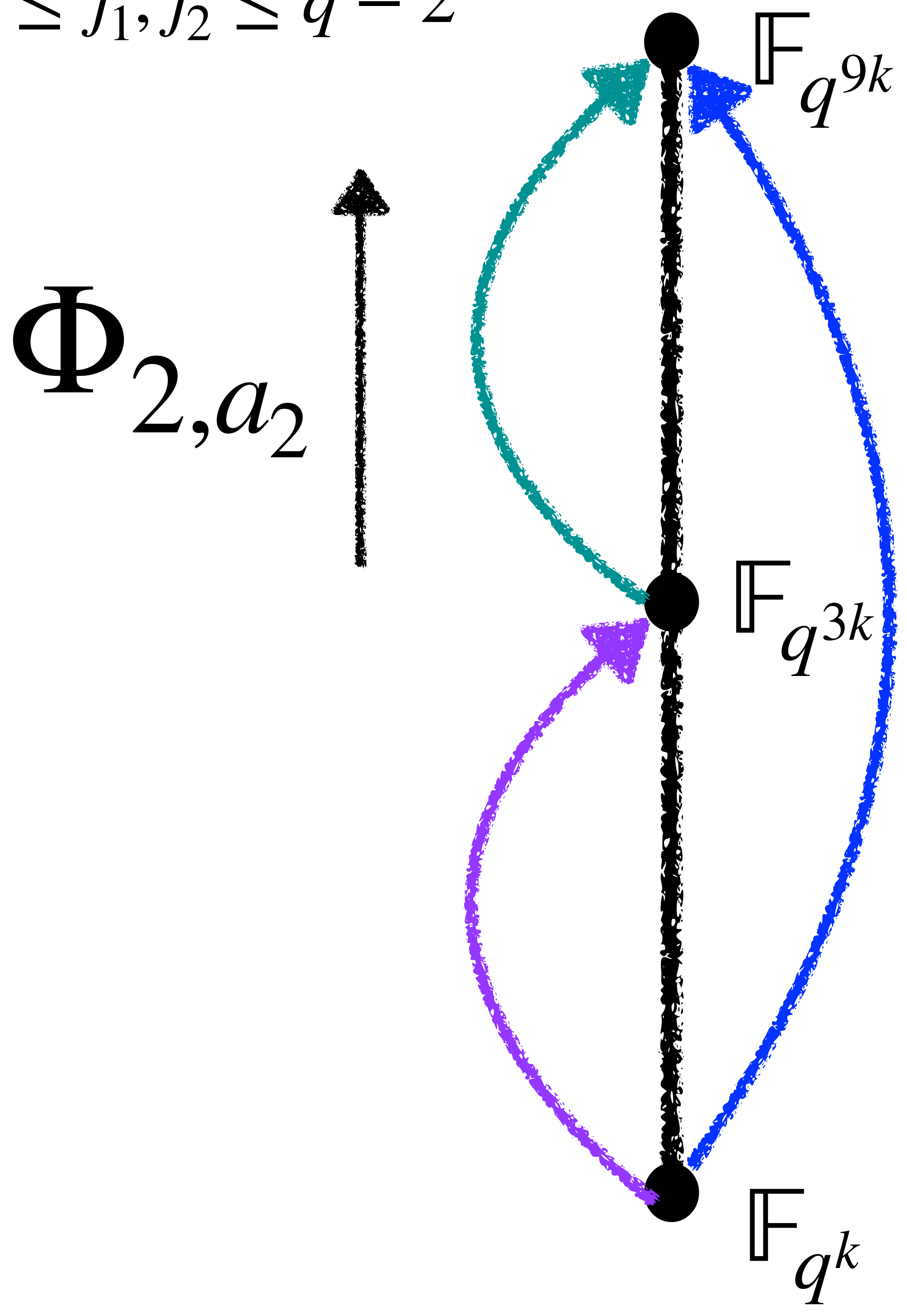
$$V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k,k)$$

$$V_{2,a_2,j_2} = \mathrm{Im}(\Phi_{2,a_2,j_2}) = \{u + \omega_2^{j_2}(u^q + ua_2)\gamma_2 : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(9k,3k)$$

$$\Phi_{2,a_2,j_2} : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{9k}}$$

injective $\quad u \mapsto u + \omega_2^{j_2}(u^q + ua_2)\gamma_2$

$\mathbb{F}_q$**-linear**



$$\mathcal{C}_1 = \cup_{0\le j_1\le q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$$

$$\mathcal{C}_2 = \cup_{0\le j_1\le q-2} \cup_{a_2\in\mathbb{F}_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2}) \subseteq \mathcal{G}_q(9k,3k)$$

$$\mathcal{C}_2 \odot \mathcal{C}_1 := \bigcup_{0\le j_1,j_2\le q-2} \bigcup_{a_2\in\mathbb{F}_{q^{3k}}} \bigcup_{a_1\in\mathbb{F}_{q^k}} \bigcup_{\alpha\in\mathbb{F}^*_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2} \odot \alpha V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(9k,k)$$

$$d(\mathcal{C}_2 \odot \mathcal{C}_1) = 2k-2$$

$$|\mathcal{C}_2 \odot \mathcal{C}_1| = |\mathcal{C}_2||\mathcal{C}_1| = q^{4k}(q^{3k}-1)(q^{9k}-1) \sim q^{16k}$$

Nested operation

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$ $\quad V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k,k)$

$\mathbb{F}_{q^{9k}} = \mathbb{F}_{q^{3k}}(\gamma_2)$ $\quad V_{2,a_2,j_2} = \mathrm{Im}(\Phi_{2,a_2,j_2}) = \{u + \omega_2^{j_2}(u^q + ua_2)\gamma_2 : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(9k,3k)$

$\omega_1 \in \mathbb{F}_{q^k}$ **Primitive element**

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$0 \leq j_1, j_2 \leq q-2$

$\Phi_{2,a_2,j_2} : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{9k}}$

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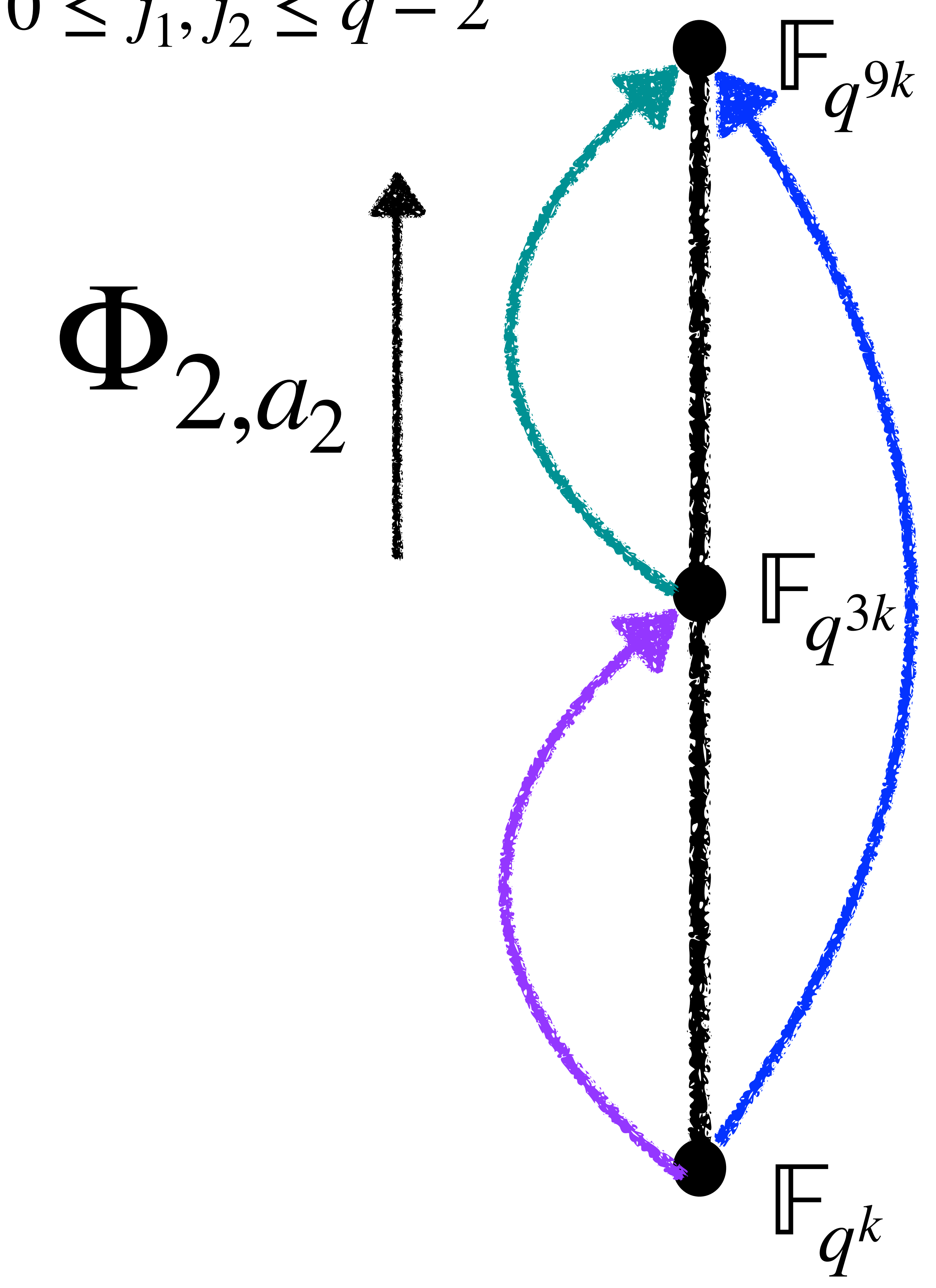
$\mathbb{F}_q$**-linear**



$\mathcal{C}_1 = \cup_{0\leq j_1\leq q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$

$\mathcal{C}_2 = \cup_{0\leq j_1\leq q-2} \cup_{a_2\in\mathbb{F}_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2}) \subseteq \mathcal{G}_q(9k,3k)$

$\mathcal{C}_2 \odot \mathcal{C}_1 := \bigcup_{0\leq j_1,j_2\leq q-2} \bigcup_{a_2\in\mathbb{F}_{q^{3k}}} \bigcup_{a_1\in}$

Johnson type bound II

$$\mathcal{A}_q(9k,2k-2,k) \leq \left\lfloor \frac{q^{9k}-1}{q^k-1} \left\lfloor \frac{q^{9k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(9k,2k-2,k) \sim q^{16k}$$

as k (or q) $\to +\infty$

$d(\mathcal{C}_2 \odot \mathcal{C}_1) = 2k-2$

$|\mathcal{C}_2 \odot \mathcal{C}_1| = |\mathcal{C}_2||\mathcal{C}_1| = q^{4k}(q^{3k}-1)(q^{9k}-1) \sim q^{16k}$

Remark

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$

$\mathbb{F}_{q^{9k}} = \mathbb{F}_{q^{3k}}(\gamma_2)$

$\omega_1 \in \mathbb{F}_{q^k}$ **Pri**

$\omega_2 \in \mathbb{F}$

0

$$V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k,k)$$

$$V_{2,a_2,j_2} = \mathrm{Im}(\Phi_{2,a_2,j_2}) = \{u + \omega_2^{j_2}(u^q + ua_2)\gamma_2 : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(9k,3k)$$

injective

$\mathbb{F}_q$-linear

$$\Phi_{2,a_2,j_2} : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{9k}}$$
$$u \mapsto u + \omega_2^{j_2}(u^q + ua_2)\gamma_2$$

$n = 9k = 3^2 k$

$$\mathcal{C}_1 = \cup_{0 \le j_1 \le q-2} \cup_{a_1 \in \mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$$

$$\mathcal{C}_2 = \cup_{0 \le j_1 \le q-2} \cup_{a_2 \in \mathbb{F}_{q^{3k}}} \mathrm{Orb}(V_{2,a_2,j_2}) \subseteq \mathcal{G}_q(9k,3k)$$

Johnson type bound II

$$\mathcal{A}_q(rk,2k-2,k) \le \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k(\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk,2k-2,k)} = 1 \Leftrightarrow r = 3$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \ge 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \ge 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \ge 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

$\mathcal{C} \subseteq \mathcal{G}_q(n',k')$ asymptotically optimal

when $n'/k' = 3$

$\bigcup \quad \bigcup \quad \bigcup$

$0 \le j_1, j_2 \le q-2 \;\; a_2 \in \mathbb{F}_{q^{3k}} \;\; a_1 \in$

Johnson type bound II

$$\mathcal{A}_q(9k,2k-2,k) \le \left\lfloor \frac{q^{9k}-1}{q^k-1} \left\lfloor \frac{q^{9k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(9k,2k-2,k) \sim q^{16k}$$

as k (or q) $\to +\infty$

$$d(\mathcal{C}_2 \odot \mathcal{C}_1) = 2k-2$$

$$\odot \mathcal{C}_1| = |\mathcal{C}_2||\mathcal{C}_1| = q^{4k}(q^{3k}-1)(q^{9k}-1) \sim q^{16k}$$

Strategy

$$n = 6k = 2 \cdot 3k$$

$$\mathscr{C}_2 = \mathrm{Orb}(V_2) = \{\beta V_2 \colon \beta \in \mathbb{F}_{q^{6k}}\} \subseteq \mathscr{G}_q(6k,3k)$$

Johnson type bound II

$$\mathscr{A}_q(2k,2k-2,k) \leq \left\lfloor \frac{q^{2k}-1}{q^k-1} \left\lfloor \frac{q^{2k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2k}$$

as k (or q) $\rightarrow +\infty$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

$\mathscr{C} \subseteq \mathscr{G}_q(n',k')$ asymptotically optimal

(within a factor of $1/2 + o_k(1)$)

when $n'/k' = 2$

$$\mathscr{C}_1 = \mathrm{Orb}(V_1) = \{\alpha V_1 \colon \alpha \in \mathbb{F}_{q^{3k}}\} \subseteq \mathscr{G}_q(3k,k)$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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Johnson type bound II

$$\mathscr{A}_q(rk,2k-2,k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\rightarrow +\infty$

$$\boxed{\lim_{k (\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk,2k-2,k)} = 1 \Leftrightarrow r = 3}$$

$\mathscr{C} \subseteq \mathscr{G}_q(n',k')$ asymptotically optimal

when $n'/k' = 3$

Strategy

$$n = 6k = 2 \cdot 3k$$

$$\mathscr{C}_2 = \mathrm{Orb}(V_2) = \{\beta V_2 \colon \beta \in \mathbb{F}_{q^{6k}}\} \subseteq \mathscr{G}_q(6k, 3k)$$

Johnson type bound II

$$\mathscr{A}_q(2k, 2k-2, k) \leq \left\lfloor \frac{q^{2k}-1}{q^k-1} \left\lfloor \frac{q^{2k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2k}$$

as k (or q) $\to +\infty$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
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$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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(within a factor of $1/2 + o_k(1)$)

when $n'/k' = 2$

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Johnson type bound II

$$\mathscr{A}_q(rk, 2k-2, k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k \text{(or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk, 2k-2, k)} = 1 \Leftrightarrow r = 3$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
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$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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$\mathscr{C} \subseteq \mathscr{G}_q(n', k')$ asymptotically optimal

when $n'/k' = 3$

Asymptotically optimal cyclic subspace codes

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$

$\mathbb{F}_{q^{6k}} = \mathbb{F}_{q^{3k}}(\gamma)$

$\omega_1 \in \mathbb{F}_{q^k}$ **Primitive element**

$0 \leq j_1 \leq q-2$

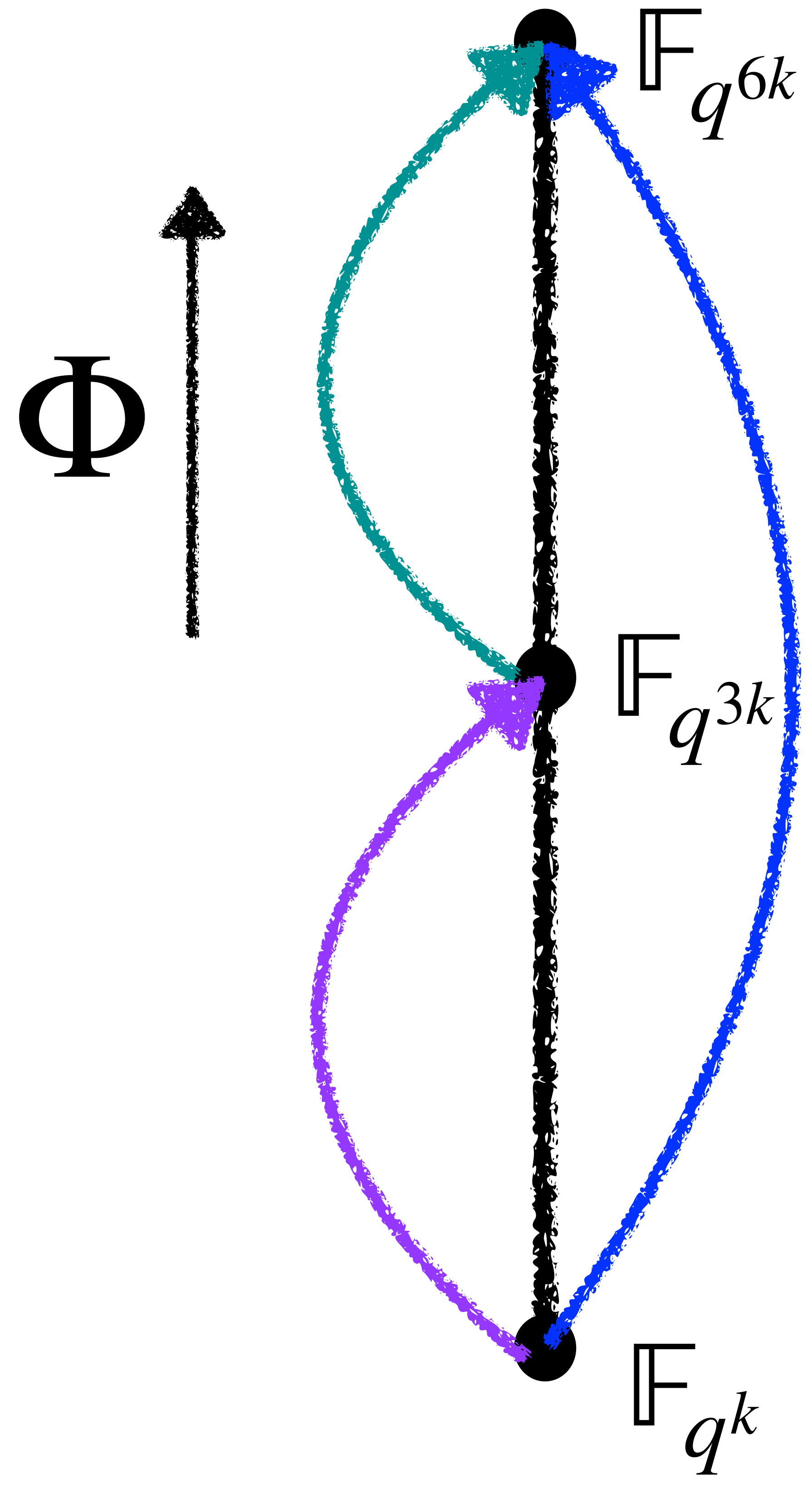
$V_{1,a_1,j_1} = \{v + \omega_1^{j_1}(v^q + a_1 v)\gamma_1 : v \in \mathbb{F}_{q^k}\} \in \mathcal{G}_q(3k,k)$

$V_2 = \mathrm{Im}(\Phi) = \{u + u^q\gamma : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(6k,3k)$

injective

$\Phi : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{6k}}$

$u \mapsto u + u^q\gamma$



$\mathscr{C}_1 = \cup_{0\leq j_1\leq q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$

$\mathscr{C}_2 = \mathrm{Orb}(V_2) = \{\beta V_2 : \beta \in \mathbb{F}_{q^{6k}}\} \subseteq \mathcal{G}_q(6k,3k)$

$$\mathscr{C}_2 \odot \mathscr{C}_1 = \bigcup_{0\leq j_1\leq q-2} \bigcup_{a_1\in\mathbb{F}_{q^k}} \bigcup_{\alpha\in\mathbb{F}^*_{q^{3k}}} \mathrm{Orb}(V_2 \odot \alpha V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(6k,k)$$

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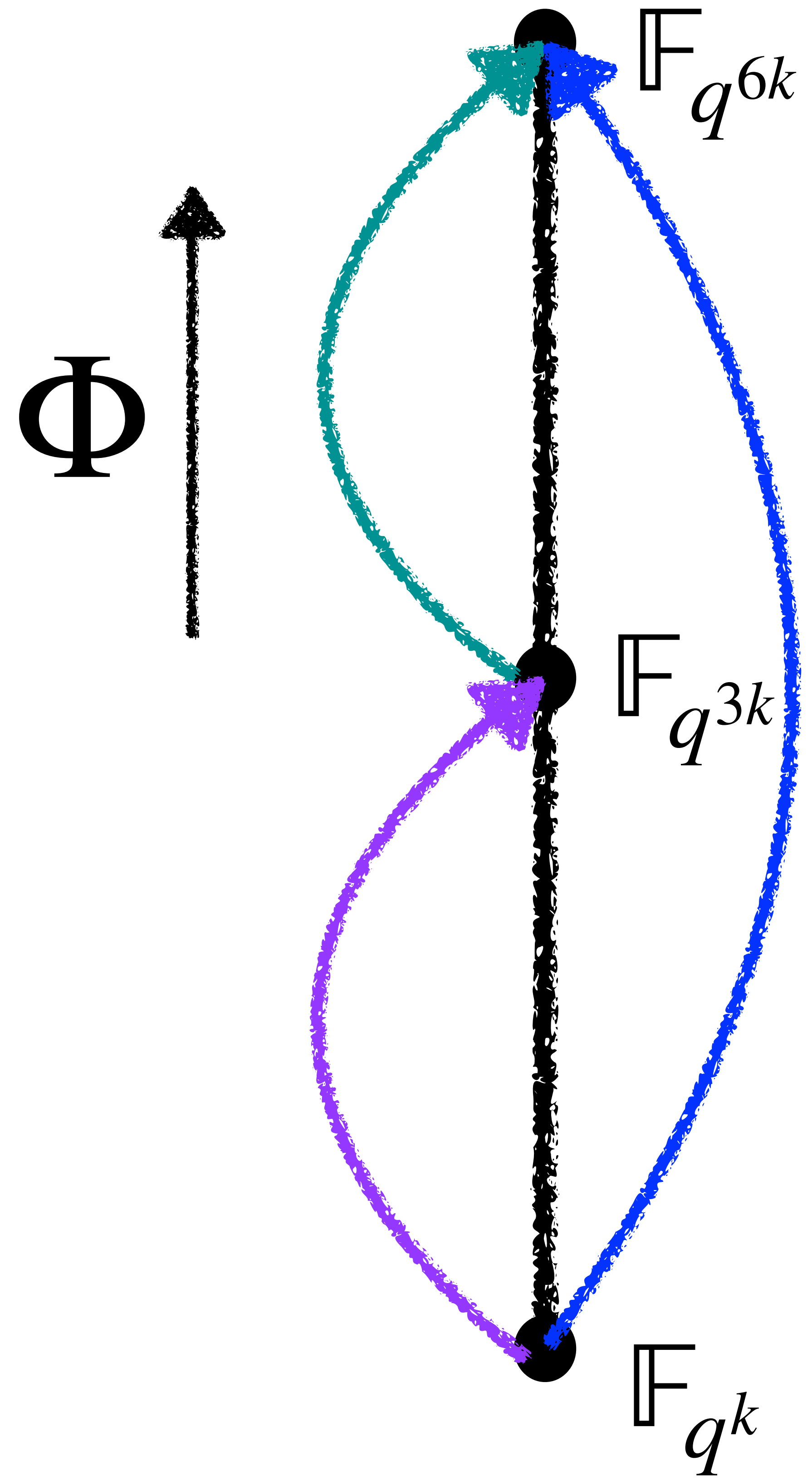

$\mathcal{C}_1 = \cup_{0 \leq j_1 \leq q-2} \cup_{a_1 \in \mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$

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$$d(\mathcal{C}_2 \odot \mathcal{C}_1) = 2k - 2$$

Asymptotically optimal cyclic subspace codes

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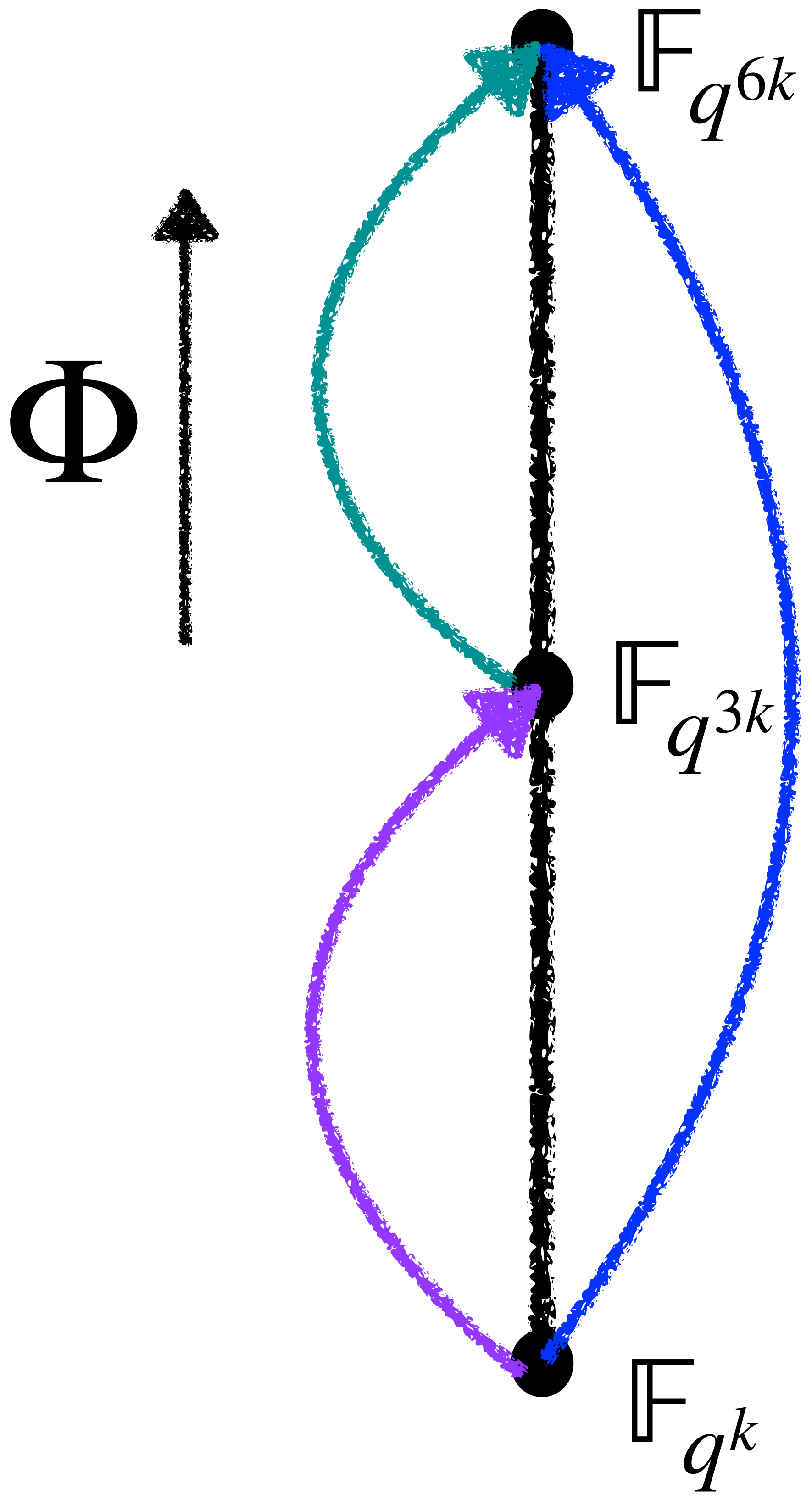

$\mathscr{C}_1 = \cup_{0\le j_1\le q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathscr{G}_q(3k,k)$

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$$\mathscr{C}_2 \odot \mathscr{C}_1 = \bigcup_{0\le j_1\le q-2}\ \bigcup_{a_1\in\mathbb{F}_{q^k}}\ \bigcup_{\alpha\in\mathbb{F}^*_{q^{3k}}} \mathrm{Orb}(V_2 \odot \alpha V_{1,a_1,j_1}) \subseteq \mathscr{G}_q(6k,k)$$

$$d(\mathscr{C}_2 \odot \mathscr{C}_1) = 2k-2$$

$$|\mathscr{C}_2 \odot \mathscr{C}_1| = |\mathscr{C}_2||\mathscr{C}_1| = \left\lfloor \frac{q-1}{2} \right\rfloor \frac{q^{6k}-1}{q-1} q^k(q^{3k}-1) \sim \frac{1}{2}q^{10k}$$

Asymptotically optimal cyclic subspace codes

$\mathbb{F}_{q^{3k}} = \mathbb{F}_{q^k}(\gamma_1)$

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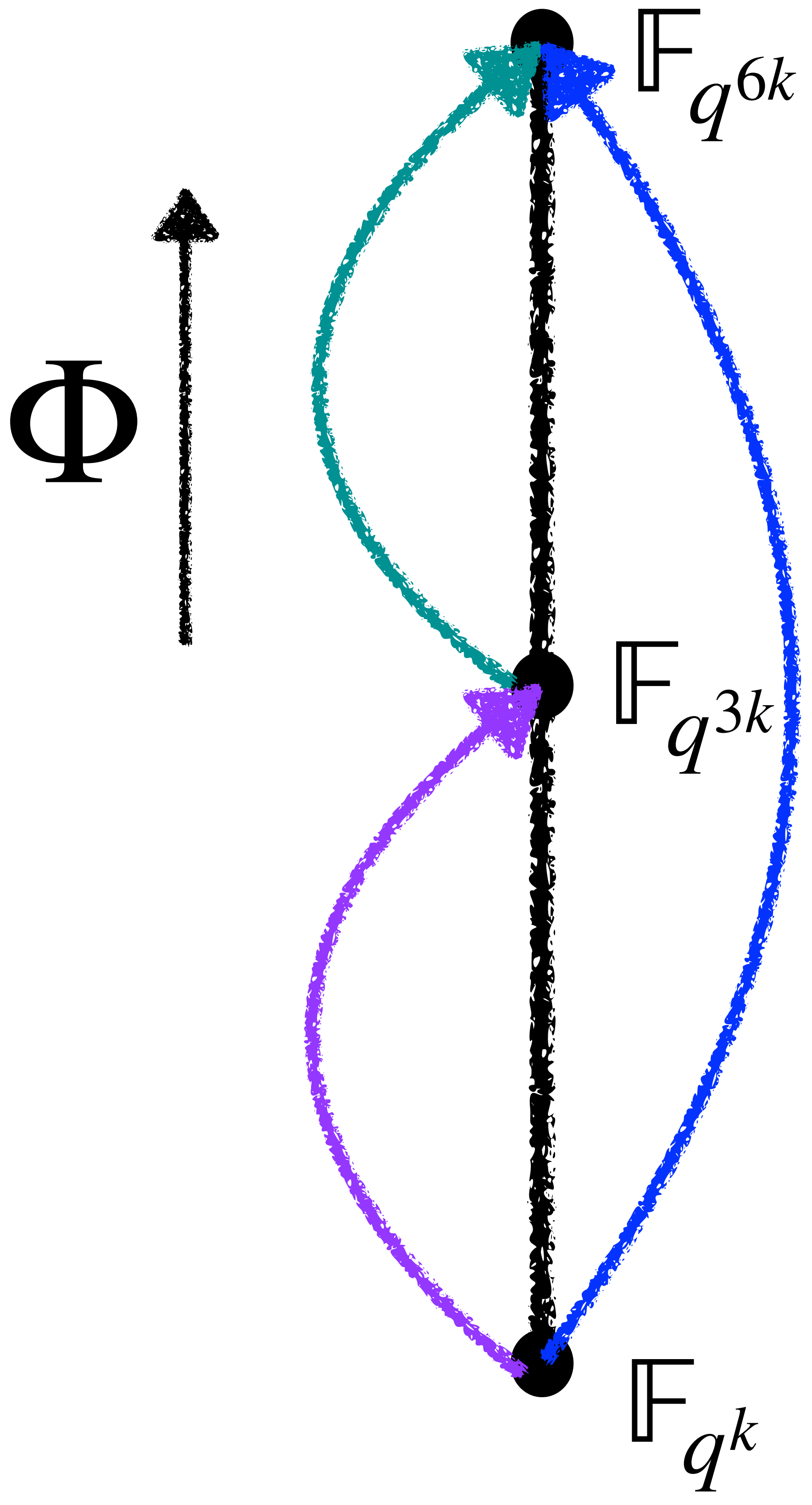
$V_2 = \mathrm{Im}(\Phi) = \{u + u^q\gamma : u \in \mathbb{F}_{q^{3k}}\} \in \mathcal{G}_q(6k,3k)$

$\Phi : \mathbb{F}_{q^{3k}} \to \mathbb{F}_{q^{6k}}$

$u \mapsto u + u^q\gamma$

injective

$\mathbb{F}_q$-linear



$\mathcal{C}_1 = \cup_{0\le j_1\le q-2} \cup_{a_1\in\mathbb{F}_{q^k}} \mathrm{Orb}(V_{1,a_1,j_1}) \subseteq \mathcal{G}_q(3k,k)$

$\mathcal{C}_2 = \mathrm{Orb}(V_2) = \{\beta V_2 : \beta \in \mathbb{F}_{q^{6k}}\} \subseteq \mathcal{G}_q(6k,3k)$

$\mathcal{C}_2 \odot \mathcal{C}_1 = \bigcup_{0\le j_1\le q-2} \bigcup_{a_1\in\mathbb{F}_{q^k}} \bigcup_{\alpha\in\mathbb{F}^*_{q^{3k}}}$

$d(\mathcal{C}_2 \odot \mathcal{C}_1) =$

Johnson type bound II

$$\mathcal{A}_q(6k,2k-2,k) \le \left\lfloor \frac{q^{6k}-1}{q^k-1} \left\lfloor \frac{q^{6k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(6k,2k-2,k) \sim q^{10k}$$

as k (or q) $\to +\infty$

$$|\mathcal{C}_2 \odot \mathcal{C}_1| = |\mathcal{C}_2||\mathcal{C}_1| = \left\lfloor \frac{q-1}{2} \right\rfloor \frac{q^{6k}-1}{q-1} q^k (q^{3k}-1) \sim \frac{1}{2} q^{10k}$$

Generalizing...

Theorem 2 (C.C., P. Santonastaso)

$$n = r_{e-1} \cdots r_1 m, e \geq 2$$

$$\mathscr{C}_1 = \bigcup_{h_1=1}^{s_1} \mathrm{Orb}(V_{1,h_1}) = \{U_{1,1}, \ldots, U_{1,t_1}\} \subseteq \mathscr{G}_q(m,k)$$

$$d(\mathscr{C}_1) = 2k - 2$$

$$\mathscr{C}_i = \bigcup_{h_i=1}^{s_i} \mathrm{Orb}(V_{i,h_i}) = \{U_{i,1}, \ldots, U_{i,t_i}\} \subseteq \mathscr{G}_q(r_{i-1} \cdots r_1 m, r_{i-2} \cdots r_1 m)$$

$$d(\mathscr{C}_i) = 2(r_{i-2} \cdots r_1 m) - 2 \qquad \forall i = 2, \ldots, e, e \geq 2$$

$$\Downarrow$$

$$\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(n,k)$$

$$d(\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1) = 2k - 2 \qquad |\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1| = |\mathscr{C}_e| \cdots |\mathscr{C}_1|$$

Consequences

$$n = 3^e k$$

$$\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(3^e k, k)$$

$$\mathscr{C}_1 \subseteq \mathscr{G}_q(3k, k)$$

$$\mathscr{C}_2 \subseteq \mathscr{G}_q(3^2 k, 3k)$$

$$\vdots$$

$$\mathscr{C}_e \subseteq \mathscr{G}_q(3^e k, 3^{e-1} k)$$

Johnson type bound II

$$\mathscr{A}_q(rk, 2k-2, k) \le \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k (\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk, 2k-2, k)} = 1 \Leftrightarrow r = 3$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$ $\square$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

$\mathscr{C} \subseteq \mathscr{G}_q(n', k')$ asymptotically optimal

when $n'/k' = 3$

Consequences

$$n = 3^e k$$

$$\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(3^e k, k)$$

$$\mathscr{C}_1 \subseteq \mathscr{G}_q(3k, k)$$

$$\mathscr{C}_2 \subseteq \mathscr{G}_q(3^2 k, 3k)$$

$$\vdots$$

$$\mathscr{C}_e \subseteq \mathscr{G}_q(3^e k, 3^{e-1} k)$$

$$|\mathscr{C}_e \odot \cdots \odot \mathscr{C}_1| \sim q^{2(3^e-1)k}$$

✓

Johnson type bound II

$$\mathscr{A}_q(rk, 2k-2, k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k (\text{or } q) \to +\infty} \frac{S(rk, k, q)}{J_q(rk, 2k-2, k)} = 1 \Leftrightarrow r = 3$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$ □	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

Johnson type bound II

$$\mathscr{A}_q(3^e k, 2k-2, k) \leq \left\lfloor \frac{q^{3^e k}-1}{q^k-1} \left\lfloor \frac{q^{3^e k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(3^e k, 2k-2, k) \sim q^{2(3^e-1)k}$$

as k (or q) $\to +\infty$

Consequences

$$n = 2^e 3^\ell k, e \geq 1, \ell \geq 0$$

$$\mathscr{C}_{e+\ell} \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(2^e 3^\ell k, k)$$

$$\mathscr{C}_1 \subseteq \mathscr{G}_q(3k, k)$$

$$\mathscr{C}_2 \subseteq \mathscr{G}_q(3^2 k, 3k)$$

$$\vdots$$

$$\mathscr{C}_{e+\ell} \subseteq \mathscr{G}_q(2^\ell 3^e k, 2^{\ell-1} 3^e k)$$

Johnson type bound II

$$\mathscr{A}_q(2k,2k-2,k) \leq \left\lfloor \frac{q^{2k}-1}{q^k-1} \left\lfloor \frac{q^{2k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2k}$$

as k (or q) $\to +\infty$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

$\mathscr{C} \subseteq \mathscr{G}_q(n', k')$ asymptotically optimal

(within a factor of $1/2 + o_k(1)$)

when $n'/k' = 2$

$\mathscr{C} \subseteq \mathscr{G}_q(n', k')$ asymptotically optimal

when $n'/k' = 3$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

Johnson type bound II

$$\mathscr{A}_q(rk,2k-2,k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\boxed{\lim_{k (\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk,2k-2,k)} = 1 \Leftrightarrow r = 3}$$

Consequences

$$n = 2^e 3^\ell k, e \geq 1, \ell \geq 0$$

$$\mathscr{C}_{e+\ell} \odot \cdots \odot \mathscr{C}_1 \subseteq \mathscr{G}_q(2^e 3^\ell k, k)$$

$$\mathscr{C}_1 \subseteq \mathscr{G}_q(3k, k)$$

$$\mathscr{C}_2 \subseteq \mathscr{G}_q(3^2 k, 3k)$$

$$\vdots$$

$$\mathscr{C}_{e+\ell} \subseteq \mathscr{G}_q(2^\ell 3^e k, 2^{\ell-1} 3^e k)$$

$$|\mathscr{C}_{e+\ell} \odot \cdots \odot \mathscr{C}_1| \sim \frac{1}{2^e} q^{2(3^\ell 2^e - 1)k}$$

Johnson type bound II

$$\mathscr{A}_q(2k, 2k-2, k) \leq \left\lfloor \frac{q^{2k}-1}{q^k-1} \left\lfloor \frac{q^{2k-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2k}$$

as k (or q) $\to +\infty$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k,k,q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k-1)(q^{4k}-1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk,k,q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk}-1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
$n = rk$, $r = 2h+1$, $h \geq 2$	$S_5(rk,k,q) = hq^k(q^k-1)^{h-1}(q^{rk}-1) + \frac{q^{rk}-1}{q^k-1}$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$

$\mathscr{C} \subseteq \mathscr{G}_q(n', k')$ asymptotically optimal

(within a factor of $1/2 + o_k(1)$)

when $n'/k' = 2$

Johnson type bound II

$$\mathscr{A}_q(3^e k, 2k-2, k) \leq \left\lfloor \frac{q^{2^e 3^\ell k}-1}{q^k-1} \left\lfloor \frac{q^{2^e 3^\ell k - 1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor := J_q(2^e 3^\ell k, 2k-2, k) \sim q^{2(2^e 3^\ell - 1)k}$$

as k (or q) $\to +\infty$

otically optimal

$= 3$

Johnson type bound II

$$\mathscr{A}_q(rk, 2k-2, k) \leq \left\lfloor \frac{q^{rk}-1}{q^k-1} \left\lfloor \frac{q^{rk-1}-1}{q^{k-1}-1} \right\rfloor \right\rfloor \sim q^{2(r-1)k}$$

as k (or q) $\to +\infty$

$$\lim_{k (\text{or } q) \to +\infty} \frac{S(rk,k,q)}{J_q(rk, 2k-2, k)} = 1 \Leftrightarrow r = 3$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k,k,q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
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$n = rk$, $r = 2h+1$, $h \geq 2$	$S_4(rk,k,q) = h\left((q^k-1)^h(q^{rk}-1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1}\right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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General construction

Theorem 3 (C.C., P. Santonastaso)

$$n = 2^e p_z^{e_z} \cdots p_1^{e_1} k \qquad 2 < p_z < \cdots < p_1 \qquad e, e_1, \ldots, e_z \geq 0$$

For k (or q) sufficiently large

$$|\mathscr{C}_{e+e_z+\ldots+e_1} \odot \cdots \odot \mathscr{C}_1| \sim \frac{1}{2^e} q^{2\prod_{i=1}^{e} p_i^{e_i} k (2^e - 1)} q^{\sum_{i=1}^{z} \left(\frac{(3p_i - 1)(p_i^{e_i} - 1)}{2(p_i - 1)} \prod_{j=1}^{i-1} p_j^{e_j} k \right)} >$$

Parameters	Size	Asymptotic behavior
$n = 2k$, $q > 2$	$S_1(2k, k, q) = \lfloor \frac{q-1}{2} \rfloor \frac{q^{2k}-1}{q-1}$	$\sim \frac{1}{2} q^{2k}$
$n = 4k$	$S_2(4k, k, q) = \lfloor \frac{q^k-2}{2} \rfloor (q^k - 1)(q^{4k} - 1)$	$\sim \frac{1}{2} q^{6k}$
$n = rk$, $r \geq 3$	$S_3(rk, k, q) = q^k \frac{q^{(\ell+1)k}-1}{q^k-1}(q^{rk} - 1)$, $\ell = (\lceil \frac{r}{2} \rceil - 2)$	$\sim q^{(\lceil \frac{r-1}{2} \rceil + r)k}$
$n = rk$, $r = 2h+1,\ h \geq 2$	$S_4(rk, k, q) = h \left((q^k - 1)^h (q^{rk} - 1) + \frac{(q^k-1)^{h-1}(q^{rk}-1)}{q-1} \right)$	$\sim q^{(\lfloor \frac{r-1}{2} \rfloor + r)k}$
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Conclusions

Before our work

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(2k,k,q)}{J_q(2k,2k-2,k)} = \frac{1}{2}$$

$n = 2k$

(Almost) Asymptotically optimal cyclic subspace codes

R. M. Roth, N. Raviv, & I. Tamo. Construction of Sidon spaces with applications to coding. *IEEE Transactions on Information Theory*, 64(6), 4412-4422, **2018**.

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$$\lim_{k(\text{or } q)\to+\infty} \frac{S(3k,k,q)}{J_q(3k,2k-2,k)} = 1$$

$n = 3k$

Asymptotically optimal cyclic subspace codes

Zhang, H., Tang, C., & Cao, X. Large optimal cyclic subspace codes. *Discrete Mathematics*, *347*(7), 114007, **2024**.

Our contribution

C.C., P. Santonastaso. Asymptotically optimal cyclic subspace codes, arXiv: 2025

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(2^e3^hk,k,q)}{J_q(2^e3^hk,2k-2,k)} = \frac{1}{2^e}, e \geq 1, h \geq 0$$

$n = 2^e3^hk$
$e \geq 1, h \geq 0$

(Almost) Asymptotically optimal cyclic subspace codes

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(3^ek,k,q)}{J_q(3^ek,2k-2,k)} = 1, e \geq 2$$

$n = 3^ek$
$e \geq 2$

Asymptotically optimal cyclic subspace codes

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Before our work

$$\lim_{k(\text{or } q)\to+\infty} \frac{S(2k,k,q)}{J_q(2k,2k-2,k)} = \frac{1}{2}$$

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$$\lim_{k(\text{or } q)\to+\infty} \frac{S(3^ek,k,q)}{J_q(3^ek,2k-2,k)} = 1,\ e \geq 2$$

$n = 3^ek$
$e \geq 2$

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Thank you for your attention!