Graphs from hyperbolic quadrics

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Introduction

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  $\mathrm{Q}^+(2n+1,q) : X_0X_1 + X_2X_3 + \ldots + X_{2n}X_{2n+1} = 0,$

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  $\mathrm{Q}^+(2n+1,q) : X_0X_1 + X_2X_3 + \ldots + X_{2n}X_{2n+1} = 0$,
- contains points, lines, planes, $\ldots$, n-dimensional projective subspaces (called *generators*),
- a finite classical polar space embedded in $\mathrm{PG}(2n+1,q)$.

The $\mathrm{NO}^+(2n+2,2)$-graph

Consider the hyperbolic quadric $\mathrm{Q}^+(2n+1,2)$. Define

- vertices as the points of $\mathrm{PG}(2n+1,2) \setminus \mathrm{Q}^+(2n+1,2)$;
- $x \sim y \iff \langle x,y \rangle$ is a tangent line to $\mathrm{Q}^+(2n+1,q)$.
- $v = 2^{2n+1} - 2^n$, $k = 2^{2n} - 1$, $\lambda = 2^{2n-1} - 2$, $\mu = 2^{2n-1} + 2^{n-1}$.

- For $n = 3$, this is an $\mathrm{srg}(120, 63, 30, 36)$, with automorphism group $\mathrm{P\Gamma O}^+(8, 2)$.
- One example of a large class of graphs, called *Fischer graphs*[1]

[1] Brouwer-Van Maldeghem, *Strongly Regular Graphs*, chapter 5

Orbital graphs

- Let G be a group acting transitively on a set Ω.
- The *orbitals* of G are its orbits on $\Omega \times \Omega$.
- Each orbital defines a directed graph on Ω.
- An $\mathrm{srg}(120, 63, 30, 36)$ arises from a rank 7 action of $\mathrm{Sym}(7)$, listed in Brouwer/Van Maldeghem, same parameters as the $\mathrm{NO}^+(8, 2)$-graph.

Question: can we construct this S_7-graph on the same vertex set as the $\mathrm{NO}^+(2n + 2, 2)$-graph?

A geometric description?

Joint work with: Sam Adriaensen, Robert Bailey, Morgan Rodgers.

- Set up $\mathrm{NO}^{+}(8,2)$ in a geometric way.
- Find copies of S_7 in $\mathrm{Aut}(\mathrm{NO}^{+}(8,2))$.
- Look at orbitals of S_7 on the vertices.
- Try out combinations of orbitals to see if we find an srg with the same parameters.
- Look for a geometrical description by exploring its adjacency relation and looking at how the group acts on the other objects.

Ovoids and spreads of $Q^+(7, q)$

Key observation

One orbit of S_7 on the generators consists of 7 mutually skew generators, another orbit a pair of mutually skew generators, together making a *spread* of $Q^+(7, q)$.

Ovoids and spreads of $\mathrm{Q}^+(7, q)$

Let $\mathcal{P}$ be a finite classical polar space.

- An ovoid of $\mathcal{P}$ is a set $\mathcal{O}$ of points such that every generator meets $\mathcal{O}$ in exactly one point.
- A spread of $\mathcal{P}$ is a set $\mathcal{S}$ of generators of $\mathcal{P}$ such that every point is contained in exactly one element of $\mathcal{S}$.

Ovoids and spreads of $\mathrm{Q}^+(7, q)$

Note

The generators of $\mathrm{Q}^+(7, q)$ come in two systems (*greek* and *latin* generators). Two generators belonging to one system meet in projective dimension -1, 1, or 3. Hence a spread consists of generators all belonging to one of the systems.

Ovoids and spreads of $Q^+(7, q)$

The quadric $Q^+(7, q)$ allows a *triality*, i.e. a map or order 3, preserving incidence, and mapping

- lines on lines,
- points on greeks,
- greeks on latins,
- latins on points.

Hence a triality maps a spread of latins on an ovoid, and an ovoid on a spread of latins.

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Corollary

Ovoids and spreads of $Q^+(7, q)$ are equivalent objects.

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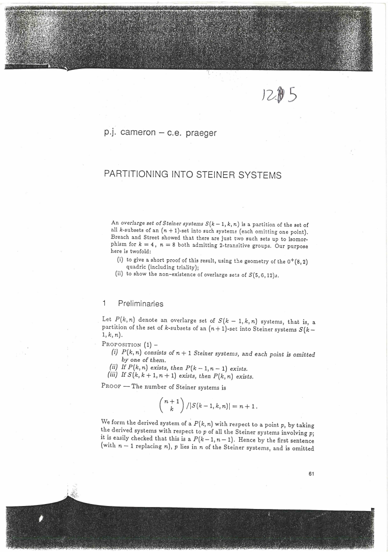
Corollary

Ovoids and spreads of $Q^+(7, q)$ are equivalent objects.

Existence or non-existence of ovoids (and hence spreads) is not settled for $Q^+(7, q)$.

Ovoids and spreads of $Q^+(7, q)$



12~~0~~5

p.j. cameron – c.e. praeger

PARTITIONING INTO STEINER SYSTEMS

An *overlarge set of Steiner systems* $S(k-1,k,n)$ is a partition of the set of all k-subsets of an $(n+1)$-set into such systems (each omitting one point). Breach and Street showed that there are just two such sets up to isomorphism for $k=4$, $n=8$ both admitting 2-transitive groups. Our purpose here is twofold:

(i) to give a short proof of this result, using the geometry of the $0^+(8,2)$ quadric (including triality);

(ii) to show the non-existence of overlarge sets of $S(5,6,12)s$.

1 Preliminaries

Let $P(k,n)$ denote an overlarge set of $S(k-1,k,n)$ systems, that is, a partition of the set of k-subsets of an $(n+1)$-set into Steiner systems $S(k-1,k,n)$.

Proposition (1) –

(i) $P(k,n)$ consists of $n+1$ Steiner systems, and each point is omitted by one of them.

(ii) If $P(k,n)$ exists, then $P(k-1,n-1)$ exists.

(iii) If $S(k,k+1,n+1)$ exists, then $P(k,n)$ exists.

Proof — The number of Steiner systems is

$$\binom{n+1}{k} / |S(k-1,k,n)| = n+1 .$$

We form the derived system of a $P(k,n)$ with respect to a point p, by taking the derived systems with respect to p of all the Steiner systems involving p; it is easily checked that this is a $P(k-1,n-1)$. Hence by the first sentence (with $n-1$ replacing n), p lies in n of the Steiner systems, and is omitted

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Ovoids and spreads of $Q^+(7, q)$

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(i) to give a short proof of this result, using the geometry of the $O^+(8, 2)$ quadric (including triality);

(ii) to show the non-existence of overlarge sets of $S(5, 6, 12)$s.

Ovoids and spreads of $Q^+(7, q)$

which maps lines to lines and preserves incidence. If p and q are points, then $p\tau \cap q\tau = \emptyset$ if and only if p and q are not perpendicular. So $Q\tau$ is a set of 9 pairwise disjoint solids, that is, a *spread* of solids. Every spread arises as the image of an ovoid under τ or τ^2. Thus the stabiliser of a spread is A_9.

More fun with spreads and ovoids

From Cameron-Praeger (1991), we know

- There is a unique spread, with automorphism group A_9.
- There are 960 copies of the unique ovoid.
- There are 2 orbits of ovoids under A_9:
  - one orbit O_1 of length 120, each ovoid having stabilizer $\mathrm{P\Gamma L}(2,8)$
  - one orbit O_2 of length 840, each ovoid having stabilizer $\mathrm{ASL}(2,3)$

The construction

- Fix the spread $\mathcal{S}$. Choose two solids π_1, π_2. The setwise stabilizer of $\{\pi_1, \pi_2\}$ in $\mathrm{Aut}(\mathcal{S})$ is S_7.
  - Any point $p \in \mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)$ determines a unique point $p_1 \in \pi_1$ and $p_2 \in \pi_2$ and vice versa.
  - Given two points $p_1 \in \pi_1$ and $p_2 \in \pi_2$, there is a unique ovoid $\mathcal{O} \in O_1$ meeting π_1 in p_1 and π_2 in p_2.
  - S_7 acts transitively on the points of $\mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)$.

The construction

- A vertex v determines a unique ovoid $\mathcal{O} \in O_1$.
- The hyperplane $v^\perp$ meets $\mathrm{Q}^+(7,2)$ in a parabolic quadric $\mathrm{Q}(6,2)$, meeting $\mathcal{O}$ in a *maximal partial ovoid* $\mathcal{O}$' of size 7.
- Each set $\mathcal{O}'' \subset \mathcal{O}'$ with $|\mathcal{O}''| = 6$ will span a 5-dimensional space Π_5, and will be a *maximal partial ovoid* of the elliptic quadric $\mathrm{Q}^-(5,2) = \Pi_5 \cap \mathrm{Q}^+(7,2)$.
- The line $\Pi_5^\perp$ will contain no points of $\mathrm{Q}^+(7,2)$.

The construction

Let v, w be two different vertices. (Recall: v determines $\mathcal{O}'$ uniquely). Then $v \sim w$ if

(a) $\langle v, w\rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\pi_1 \cup \pi_2$; or
(b) $\langle v, w\rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\mathcal{O}'$; or
(c) $\langle v, w\rangle$ is a line skew to $\mathrm{Q}^+(7,2)$ and $\langle v, w\rangle^\perp$ does not meet $\mathcal{O}'$ in 6 points.

The construction

Let v, w be two different vertices. (Recall: v determines $\mathcal{O}'$ uniquely). Then $v \sim w$ if

(a) $\langle v, w \rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\pi_1 \cup \pi_2$;
This gives 14 adjacencies;

(b) $\langle v, w \rangle$ is tangent to $\mathrm{Q}^+(7,2)$ and meets $\mathrm{Q}^+(7,2)$ in a point of $\mathcal{O}'$;
This gives 7 adjacencies;

(c) $\langle v, w \rangle$ is a line skew to $\mathrm{Q}^+(7,2)$ and $\langle v, w \rangle^\perp$ does **not** meet $\mathcal{O}'$ in 6 points.
There are 28 lines on v skew to $\mathrm{Q}^+(7,2)$. Because there are exactly 7 sets $\mathcal{O}''$ of size 6, there are exactly 7 lines l_i on v skew to $\mathrm{Q}^+(7,2)$ such that $l_i^\perp$ meets $\mathcal{O}'$ in such a set $\mathcal{O}''$. So there are exactly 21 skew lines satisfying the condition, each line contains 2 vertices adjacent to v, hence 42 adjacencies.

The graph $\mathcal{G}_n(q)$

Consider the hyperbolic quadric $\mathrm{Q}^+(2n+1,q)$. Fix a generator Π. Define

- vertices as the points of $\mathrm{Q}^+(2n+1,q) \setminus \Pi$;
- $x \sim_1 y \iff \langle x, y \rangle$ is a secant line to $\mathrm{Q}^+(2n+1,q)$;
- $x \sim_2 y \iff \langle x, y \rangle \subset q^+(2n+1,q)$ and $\langle x, y \rangle$ meet Π in a point.
- $\sim = \sim_1 \cup \sim_2$.

The graph $\mathcal{G}_n(q)$

The graph $\mathcal{G}_n(q)$ is a strongly regular graph with parameters
$v = \frac{(q^n+1)(q^{n+1}-1)}{q-1} - \frac{q^{n+1}-1}{q-1} = \frac{q^n(q^{n+1}-1)}{q-1}$, $k = q^{2n} - 1$,
$\lambda = q^{2n-1}(q-1) - 2$ and $\mu = (q^{2n-1} + q^{n-1})(q-1)$.

The graph $\mathcal{G}_n(q)$

The graphs $\mathcal{G}_n(2)$ and $\mathrm{NO}^+(2n+2, 2)$ have the same parameters.

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Romaniello and Smaldore

The graphs $\mathcal{G}_3(2)$ and $\mathrm{NO}^+(6, 2)$ are isomorphic

Cliques of $NO^+(8,2)$

Delsarte clique bound

Let Γ be a strongly regular graph with regularity k and smallest eigenvalue λ. Then the size of a clique in Γ is at most $1 - \frac{k}{\lambda}$.

The size of a clique in $NO^+(8,2)$ and $\mathcal{G}_3(2)$ is at most 8.

Brouwer-Van Maldeghem

All maximal cliques of $NO^+(2n+2,2)$ appear as follows. Let π be an n-space of $\mathrm{PG}(2n+1,q)$ meeting $\mathrm{Q}^+(2n+1,q)$ in an $(n-1)$-space. The 2^n points of $\pi \setminus \mathrm{Q}^+(2n+1,q)$ are a clique of size 2^n, necessarily maximal, and no other maximal cliques exist in $NO^+(2n+2,2)$.

Cliques of $\mathcal{G}_3(2)$

Joint work with Antonio Cossidente, Giuseppe Marino, Francesco Pavese, Valentino Smaldore

# of cliques	**Size**	**Adjacencies**	**Geometric description**
10752	5	1	$Q^-(3,2) \subseteq Q^+(7,2)$ not meeting Π_∞
960	8	1	Cameron-Praeger ovoid [5]
15	8	2	Generators of $Q^+(7,2)$ meeting Π_∞ in a plane
840	8	mixed	Cones $PQ^-(3,2)$, $P \in \Pi_\infty$, meeting Π_∞ in a line
210	8	mixed	Cones $\ell Q^+(1,2)$, $\ell \in \Pi_\infty$, meeting Π_∞ in ℓ