Binary code generated by the hyperbolic quadrics of $W(2n-1, q)$, q even

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Polarities in projective spaces

A polarity in $\mathrm{PG}(n, q)$, $n \geq 2$, is a permutation ζ of the set of subspaces reversing the inclusion relation.

- orthogonal polarities
- Hermitean polarities
- pseudo-polarities
- symplectic polarities

Symplectic polarities

If $\mathrm{PG}(n,q) = \mathrm{PG}(V)$ with $n = \mathrm{dim}(V) - 1$ odd and f is a nondegenerate alternating bilinear form on V, then

$$PG(U)^{\zeta} = \mathrm{PG}(U^{\perp}),$$

where

$$U^{\perp} = \{v \in V \mid f(v,u) = 0, \forall u \in U\}.$$

For symplectic polarities, we have $x \in x^{\zeta}$ for every point x of $\mathrm{PG}(n,q)$.

Symplectic polar spaces

Let ζ be a symplectic polarity of $\mathrm{PG}(2n-1,q)$.

A subspace π of $\mathrm{PG}(2n-1,q)$ is totally isotropic with respect to ζ if $\pi \subseteq \pi^\zeta$. The set of totally isotropic subspaces with respect to ζ defines a so-called symplectic polar space $W(2n-1,q)$:

- lines of $W(2n-1,q)$ are the totally isotropic lines;
- the other lines of $\mathrm{PG}(2n-1,q)$ are called hyperbolic lines.

Ovoids of $\mathrm{PG}(3, q)$

An ovoid of $\mathrm{PG}(3, q)$ is a nonempty set O points with the property that through each point x, there is a unique plane π_x for which the following hold:

(1) $\pi_x \cap O = \{x\}$;

(2) every line through x not contained in π_x intersects O in exactly two points.

For q odd, every ovoid is a so-called elliptic quadric.

If q is even, then there exists a unique symplectic polarity ζ of $\mathrm{PG}(3, q)$ such that $\pi_x = x^\zeta$ for every point x of O.

If $W(3, q)$ is the symplectic polar space associated to ζ, then O is called an ovoid of $W(3, q)$.

Nonsingular quadrics

Consider the n-dimensional projective space $\mathrm{PG}(n, q)$. A quadric of $\mathrm{PG}(n, q)$ is set of points whose homogenous coordinates $(X_0, X_1, \ldots, X_n)$ satisfy an equation of the form

$$\Sigma_{0 \leq i \leq j \leq n} a_{ij} X_i X_j = 0.$$

Such a quadric Q is called nonsingular if there are no points of Q every line through which intersects Q in either 1 or $q + 1$ points.

Types of nonsingular quadrics

- parabolic quadric $Q(2m, q)$ in $\mathrm{PG}(2m, q)$ (Witt index m)

$$X_0^2 + X_1X_2 + \cdots + X_{2m-1}X_{2m} = 0.$$

- hyperbolic quadric $Q^+(2m-1, q)$ in $\mathrm{PG}(2m-1, q)$ (Witt index m)

$$X_0X_1 + X_2X_3 + \cdots + X_{2m-2}X_{2m-1} = 0.$$

- elliptic quadric $Q^-(2m-1, q)$ in $\mathrm{PG}(2m-1, q)$ (Witt index $m-1$)

$$f(X_0, X_1) + X_2X_3 + \cdots + X_{2m-2}X_{2m-1} = 0,$$

with f irreducible quadratic homogeneous polynomial.

Associated polarities

Let Q be a nonsingular quadric in $\mathrm{PG}(N, q)$ with q odd.

- If q is odd, then there exists a unique orthogonal polarity ζ such that $x \in Q$ if and only if $x \in x^\zeta$. For every point $x \in Q$, x^ζ is the hyperplane which is tangent to Q in the point x.
- For the quadrics $Q^+(2n-1, q)$ and $Q^-(2n-1, q)$ with q even, there exists a unique symplectic polarity ζ such that for every point $x \in Q$, x^ζ is the hyperplane which is tangent to Q in the point x.

Hyperbolic and elliptic quadrics of $W(2n-1,q)$, q even

Let ζ be a symplectic polarity of $\mathrm{PG}(2n-1,q)$, $n \geq 2$ and q even.

Let $W(2n-1,q)$ be the associated symplectic polar space.

Every hyperbolic/elliptic quadric for which ζ is the associated symplectic polarity is called a hyperbolic/elliptic quadric of $W(2n-1,q)$.

The codes under consideration

Let $\mathcal{H}$ be the binary code generated by the (characteristic vectors of the) hyperbolic quadrics of $W(2n-1,q)$, $n \geq 2$ and q even.

Let $\mathcal{H}^{\perp}$ be its dual code.

Theorem (Sastry and Sin, 2001)

$\mathcal{H}$ coincides with code generated by the elliptic quadrics of $W(2n-1,q)$.

Main results, I

Theorem (Das, DB, Sahoo and Sastry)

The minimum weight of $\mathcal{H}^\perp$ is $q+1$. The codewords of $\mathcal{H}^\perp$ of minimum weight are the hyperbolic lines of $W(2n-1,q)$.

Theorem (Das, DB, Sahoo and Sastry)

The minimum weight of $\mathcal{H}$ is $\frac{(q^{n-1}-1)(q^n+1)}{q-1}$, and the following hold:

1. *If $n=2$, then the codewords of $\mathcal{H}$ of minimum weight are the ovoids of $W(3,q)$ contained in $\mathcal{H}$.*
2. *If $n \geq 3$, then the codewords of $\mathcal{H}$ of minimum weight are the elliptic quadrics of $W(2n-1,q)$.*

Main results, II

Theorem (Das, DB, Sahoo and Sastry)

The maximum weight of $\mathcal{H}^{\perp}$ is $q^{2n-1} + q^{2n-2} + \cdots + q^2$. The codewords of $\mathcal{H}^{\perp}$ of maximum weight are the complements of the lines of $W(2n-1, q)$.

Theorem (Das, DB, Sahoo and Sastry)

The maximum weight of $\mathcal{H}$ is q^{2n-1} if $q \geq 4$, and is $2^{2n-1} + 2^{n-1} - 1$ if $q = 2$.

1. *If $q \geq 4$, then the codewords of $\mathcal{H}$ of maximum weight are the complements of the hyperplanes of $\mathrm{PG}(2n-1, q)$;*
2. *If $q = 2$, then the codewords of $\mathcal{H}$ of maximum weight are the hyperbolic quadrics of $W(2n-1, 2)$.*

Auxiliary results and tools, I

Let $\mathcal{L}$ be a set of lines of $\mathrm{PG}(m,q)$. A $\mathcal{L}$-blocking set is a set of points of $\mathrm{PG}(m,q)$ meeting each line of $\mathcal{L}$.

Theorem (Das, DB, Sahoo and Sastry)

Let $\mathcal{L}$ be the set of all hyperbolic lines of $W(2n-1,q)$ and let B be an $\mathcal{L}$-blocking set in $\mathrm{PG}(2n-1,q)$ of minimum size. Then the following hold:

(1) *If $q \geq 3$, then $|B| = \frac{q^{2n-1}-1}{q-1}$ and B is a hyperplane of $\mathrm{PG}(2n-1,q)$.*

(2) *If $q = 2$, then $|B| = 2^{2n-1} - 2^{n-1}$ and B is the complement of a hyperbolic quadric of $W(2n-1,2)$.*

Auxiliary results and tools, II

Theorem (Metsch, 1999)

Let $\mathcal{L}$ be the set of all lines of $W(2n-1,q)$ and let B be an $\mathcal{L}$-blocking set in $\mathrm{PG}(2n-1,q)$ of minimum size. Then $|B| = \frac{(q^{n-1}-1)(q^n+1)}{q-1}$ and the following hold:

(1) *if $n = 2$, then B is an ovoid of $W(3,q)$;*

(2) *if $n \geq 3$, then B is an elliptic quadric of $W(2n-1,q)$.*

Auxiliary results and tools, III

Theorem (Das, DB, Sahoo and Sastry)

The following are equivalent for a set X of $q+1$ points of $W(2n-1,q)$, q even:

(a) X *is a hyperbolic line of* $W(2n-1,q)$;

(b) $|X \cap E| \in \{0,2\}$ *for every elliptic quadric* E *of* $W(2n-1,q)$;

(c) $|X \cap E| \neq 1$ *for every elliptic quadric* E *of* $W(2n-1,q)$.

Auxiliary results and tools, IV

- The orbits of $PSp(2n, q)$ on the (un)ordered triples of three points, every two of which lie on a hyperbolic line.
- The notion of a rosette of elliptic quadrics of $W(2n-1, q)$: these elliptic quadrics mutually intersect in a common point x and have the same tangent hyperplane Π in x. They form a single orbit of the group of symplectic transvections with center x and axis Π.