Cameron-Liebler sets

in the

Klein quadric $Q^+(5, q)$

Jozefien D'haeseleer

Joint work with Leo Storme and Jonathan Mannaert

Finite Geometries 2025
7th Irsee conference

GHENT UNIVERSITY

Outline

1. Cameron-Liebler sets in finite classical polar spaces
2. Cameron-Liebler sets in the Klein quadric $Q^+(5, q)$

1. Cameron-Liebler sets in finite classical polar spaces

2. Cameron-Liebler sets in the Klein quadric $Q^+(5, q)$

4 Finite classical polar spaces

- Hyperbolic quadric $Q^+(2n+1, q)$
- Elliptic quadric $Q^-(2n+1, q)$
- Parabolic quadric $Q(2n, q)$
- Hermitian variety $H(n, q^2)$
- Symplectic polar space $W(2n+1, q)$

5 Combinatorial definition

Definition

Let $\mathcal{P}$ be a finite classical polar space of rank d, let $\mathcal{L}$ be a set of generators in $\mathcal{P}$. Then $\mathcal{L}$ is a degree 1 CL set of generators in $\mathcal{P}$ if and only if the number of elements of $\mathcal{L}$ meeting a generator π in a codimension 1-space equals

$$\begin{cases} x - 1 + q^e \frac{q^{d-1}-1}{q-1} & \text{if } \pi \in \mathcal{L} \\ x & \text{if } \pi \notin \mathcal{L}. \end{cases}$$

6 Combinatorial definition

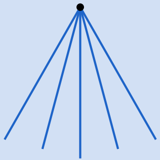
Definition

Let $\mathcal{P}$ be a finite classical polar space of rank d, let $\mathcal{L}$ be a set of generators in $\mathcal{P}$ and let i be any* integer in $\{1, \dots, d-1\}$. Then $\mathcal{L}$ is a degree 1 CL set of generators in $\mathcal{P}$ if and only if the number of elements of $\mathcal{L}$ meeting a generator π in a codimension i-space equals

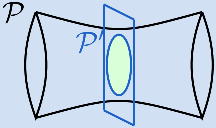
$$\begin{cases} \left((x-1)\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} + q^{i+e-1}\begin{bmatrix} d-1 \\ i \end{bmatrix}\right) q^{\frac{(i-1)(i-2)}{2}+(i-1)e} & \text{if } \pi \in \mathcal{L} \\ x\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} q^{\frac{(i-1)(i-2)}{2}+(i-1)e} & \text{if } \pi \notin \mathcal{L} \end{cases}$$

7 Some trivial examples

Point-pencil



Embedded polar space



Question

Are there non-trivial examples of degree 1 Cameron-Liebler sets in finite classical polar spaces?

See previous talk by Morgan Rodgers.

Question

Are there non-trivial examples of degree 1 Cameron-Liebler sets in finite classical polar spaces?

See previous talk by Morgan Rodgers.

9 Classification results

Theorem (M. De Boeck, J.D., M. Rodgers, L. Storme, A. Švob)

If $\mathcal{L}$ is a (degree 1) CL set of $\mathcal{P}$ with parameter x, then $x \in \mathbb{N}$.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x. If $x \leq q^{e-1} + 1$, then $\mathcal{L}$ is the union of x point-pencils whose vertices are pairwise non-collinear or $x = q^{e-1} + 1$ and $\mathcal{L}$ is the set of generators in an embedded polar space.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{P}$ be the polar space $\mathcal{W}(5, q)$ or $\mathcal{Q}(6, q)$ and let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x, $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$. Then $\mathcal{L}$ is a union of embedded polar spaces $Q^+(5, q)$ and point-pencils.

9 Classification results

Theorem (M. De Boeck, J.D., M. Rodgers, L. Storme, A. Švob)

If $\mathcal{L}$ is a (degree 1) CL set of $\mathcal{P}$ with parameter x, then $x \in \mathbb{N}$.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x. If $x \leq q^{e-1} + 1$, then $\mathcal{L}$ is the union of x point-pencils whose vertices are pairwise non-collinear or $x = q^{e-1} + 1$ and $\mathcal{L}$ is the set of generators in an embedded polar space.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{P}$ be the polar space $\mathcal{W}(5, q)$ or $\mathcal{Q}(6, q)$ and let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x, $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$. Then $\mathcal{L}$ is a union of embedded polar spaces $Q^+(5, q)$ and point-pencils.

9 Classification results

Theorem (M. De Boeck, J.D., M. Rodgers, L. Storme, A. Švob)

If $\mathcal{L}$ is a (degree 1) CL set of $\mathcal{P}$ with parameter x, then $x \in \mathbb{N}$.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x. If $x \leq q^{e-1} + 1$, then $\mathcal{L}$ is the union of x point-pencils whose vertices are pairwise non-collinear or $x = q^{e-1} + 1$ and $\mathcal{L}$ is the set of generators in an embedded polar space.

Theorem (M. De Boeck, J.D.)

Let $\mathcal{P}$ be the polar space $\mathcal{W}(5, q)$ or $\mathcal{Q}(6, q)$ and let $\mathcal{L}$ be a degree 1 CL set of $\mathcal{P}$ with parameter x, $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$. Then $\mathcal{L}$ is a union of embedded polar spaces $Q^+(5, q)$ and point-pencils.

1 Cameron-Liebler sets in finite classical polar spaces

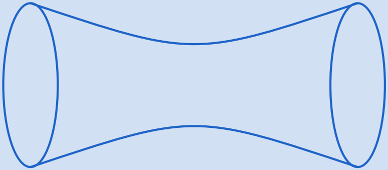
2 Cameron-Liebler sets in the Klein quadric $Q^+(5, q)$

11 The hyperbolic quadric $Q^+(5, q)$

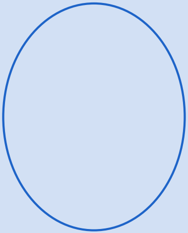
- A non-singular quadric with standard equation $X_0X_1 + X_2X_3 + X_4X_5 = 0$.
- Contains points, lines and planes.
- The generators (planes), of $Q^+(5, q)$ can be partitioned into two classes, often called the class of the *Latin* generators and the class of the *Greek* generators.
  - Two generators Π_1 and Π_2 of the hyperbolic quadric $Q^+(5, q)$ are equivalent if and only if they are equal or intersect in a point.

The Klein Correspondence

$Q^+(5, q)$

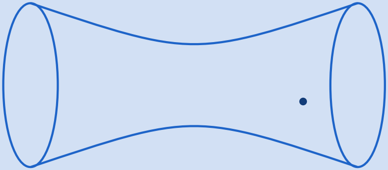


$\mathrm{PG}(3, q)$

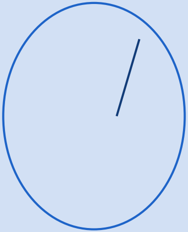


12 The Klein Correspondence

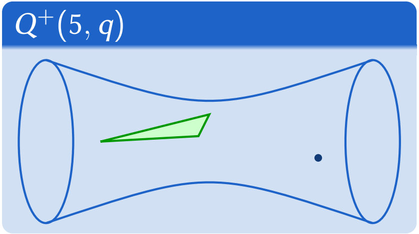
$Q^+(5, q)$

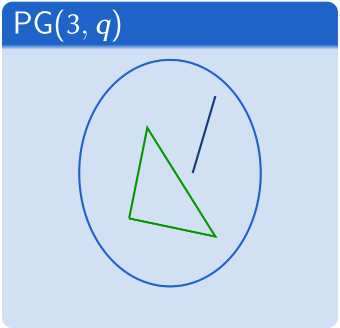


$\mathrm{PG}(3, q)$



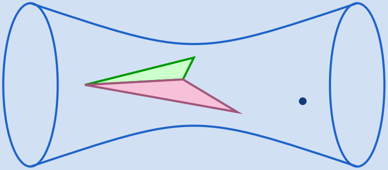
The Klein Correspondence



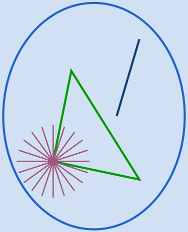


12 The Klein Correspondence

$Q^+(5, q)$



$\mathrm{PG}(3, q)$



13 The Klein correspondence

$\mathbf{PG(3, q)}$	$\boldsymbol{Q^+(5, q)}$
Line	Point
Two intersecting lines	Two points, contained in a common line of $Q^+(5, q)$
The set of lines through a fixed point P and in a fixed plane π, with $P \in \pi$	Line
The set of lines in a fixed plane	Greek plane
The set of lines through a fixed point	Latin plane

14 CL sets in the Klein quadric $Q^+(5, q)$

Definition

Let $\mathcal{Q} = Q^+(5, q)$ be the Klein quadric, let $\mathcal{L}$ be a set of generators in $\mathcal{Q}$. Then $\mathcal{L}$ is a CL set of generators in $\mathcal{Q}$ if and only if the number of elements of $\mathcal{L}$ meeting a plane π in a line equals

$$\begin{cases} x+q & \text{if } \pi \in \mathcal{L} \\ x & \text{if } \pi \notin \mathcal{L}. \end{cases}$$

Moreover $|\mathcal{L}| = 2x(q+1)$, and $\mathcal{L}$ consist of $x(q+1)$ Latin and $x(q+1)$ Greek planes.

14 CL sets in the Klein quadric $Q^+(5, q)$

Definition

Let $\mathcal{Q} = Q^+(5, q)$ be the Klein quadric, let $\mathcal{L}$ be a set of generators in $\mathcal{Q}$. Then $\mathcal{L}$ is a CL set of generators in $\mathcal{Q}$ if and only if the number of elements of $\mathcal{L}$ meeting a plane π in a line equals

$$\begin{cases} x + q & \text{if } \pi \in \mathcal{L} \\ x & \text{if } \pi \notin \mathcal{L}. \end{cases}$$

Moreover $|\mathcal{L}| = 2x(q + 1)$, and $\mathcal{L}$ consist of $x(q + 1)$ Latin and $x(q + 1)$ Greek planes.

15 CL sets in the Klein quadric $Q^+(5, q)$

CL sets under the Klein correspondence

Suppose that P_0 is a set of points and P_2 is a set of planes in $\mathrm{PG}(3, q)$, respectively. Then the following statements are equivalent:

1. The set of generators in $Q^+(5, q)$ derived from P_0 and P_2, using the Klein correspondence is a Cameron-Liebler set of parameter x.
2. The following two properties are valid.
   - Every plane of $\mathrm{PG}(3, q)$ contains x or $q + x$ points of P_0, and the planes of P_2 are the planes containing $q + x$ points of P_0.
   - Every point of $\mathrm{PG}(3, q)$ lies in x or $q + x$ planes of P_2, and the points of P_0 are the points lying in $q + x$ planes of P_2.

Examples

CL sets coming from partial line spreads in $\mathrm{PG}(3, q)$

- Let S be a maximal partial spread of size $q^2 + 1 - x$.
- The set of holes P_0 and the set of planes P_2, not containing a line of S, both have size $x(q+1)$.
- $P_0 \cup P_2$ gives a CL set of parameter x in $Q^+(5, q)$.

Examples

CL sets coming from partial line spreads in $\mathrm{PG}(3, q)$

- Let S be a maximal partial spread of size $q^2 + 1 - x$.
- The set of holes P_0 and the set of planes P_2, not containing a line of S, both have size $x(q+1)$.
- $P_0 \cup P_2$ gives a CL set of parameter x in $Q^+(5, q)$.

Examples

CL sets coming from partial line spreads in $\mathrm{PG}(3, q)$

- Let S be a maximal partial spread of size $q^2 + 1 - x$.
- The set of holes P_0 and the set of planes P_2, not containing a line of S, both have size $x(q + 1)$.
- $P_0 \cup P_2$ gives a CL set of parameter x in $Q^+(5, q)$.

Examples

CL sets coming from Bear subgeometries in $PG(3, q^2)$

- Set of points and planes of the Baer subgeometry $PG(3, q)$ in $PG(3, q^2)$.
- Gives CL set $\mathcal{L}$ of parameter $q + 1$ in $Q^+(5, q^2)$.
- Then $\mathcal{L}$ is a sub hyperbolic quadric $Q^+(5, q)$ in $Q^+(5, q^2)$ and is called *Baer subgeometry type*.

Examples

CL sets coming from Bear subgeometries in $PG(3, q^2)$

- Set of points and planes of the Baer subgeometry $PG(3, q)$ in $PG(3, q^2)$.
- Gives CL set $\mathcal{L}$ of parameter $q + 1$ in $Q^+(5, q^2)$.
- Then $\mathcal{L}$ is a sub hyperbolic quadric $Q^+(5, q)$ in $Q^+(5, q^2)$ and is called *Baer subgeometry type*.

Examples

CL sets coming from Bear subgeometries in $PG(3, q^2)$

- Set of points and planes of the Baer subgeometry $PG(3, q)$ in $PG(3, q^2)$.
- Gives CL set $\mathcal{L}$ of parameter $q + 1$ in $Q^+(5, q^2)$.
- Then $\mathcal{L}$ is a sub hyperbolic quadric $Q^+(5, q)$ in $Q^+(5, q^2)$ and is called *Baer subgeometry type*.

CL sets coming from linear sets

Theorem (J.D., J. Mannaert, L. Storme)

There exist Cameron-Liebler sets on the Klein quadric $Q^+(5, q^t)$ with parameter $x = \frac{q^t-1}{q-1}$, arising from scattered $\mathbb{F}_q$-linear sets of rank $\frac{rt}{2}$.

Classification results in $Q^+(5, q)$

Classification results

- If $\mathcal{L}$ is a (degree 1) Cameron-Liebler set of $\mathcal{P}$ with parameter x, then $x \in \mathbb{N}$.
- If $\mathcal{L}$ is a (degree 1) Cameron-Liebler set of $\mathcal{P}$ with parameter 1, then $\mathcal{L}$ is a point-pencil.

New classification result

Theorem (J.D., J. Mannaert, L. Storme)

Every Cameron-Liebler set $\mathcal{L}$ on the Klein quadric, with parameter x satisfying $1 \leq x < \sqrt{q} + 1$, is the union of x point-pencils, defined by x points pairwise non-collinear on the Klein quadric.

Link with holes of maximal partial spread

Method

Holes of maximal partial spreads in $\mathrm{PG}(3, q)$.

- Link with non-trivial blocking sets in $\mathrm{PG}(2, q)$.
- Characterisations of these blocking sets, and hence, of the sets of holes were found.
  - The proof only uses combinatorial properties.
  - The proof can be repeated in the context of CL sets!

Link with holes of maximal partial spread

Method

Holes of maximal partial spreads in $\mathrm{PG}(3, q)$.

- Link with non-trivial blocking sets in $\mathrm{PG}(2, q)$.
- Characterisations of these blocking sets, and hence, of the sets of holes were found.
  - The proof only uses combinatorial properties.
  - The proof can be repeated in the context of CL sets!

Link with holes of maximal partial spread

Method

Holes of maximal partial spreads in $\mathrm{PG}(3, q)$.

- Link with non-trivial blocking sets in $\mathrm{PG}(2, q)$.
- Characterisations of these blocking sets, and hence, of the sets of holes were found.
    - The proof only uses combinatorial properties.
    - The proof can be repeated in the context of CL sets!

Classification results

Theorem (J. D., J. Mannaert, L. Storme)

1. Let $p = p_0^h$, $p_0 \geq 7$ a prime, $h \geq 1$ odd.

   Let $\mathcal{L}$ be a CL set of generators on $Q^+(5, p^3)$, with $x \leq \delta_0$, then $\mathcal{L}$ is the union of of disjoint sets of the following types

   - point-pencils,
   - CL sets of projected $\mathrm{PG}(5, p)$ type.

2. Let $p = p_0^h$, $p_0 \geq 7$ a prime, $h > 1$ even.

   Let $\mathcal{L}$ be a CL set of generators on $Q^+(5, p^3)$, with $x \leq \delta_0$, then $\mathcal{L}$ is the union of disjoint sets of the following types

   - point-pencils,
   - CL sets of Baer subgeometry type,
   - CL sets of projected $\mathrm{PG}(5, q)$ type.

Thank you very much for your attention.