Codes from the Point–Hyperplane Geometry of $\mathrm{PG}(V)$

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Notation

- $\mathbb{F}_q$: finite fields with q elements;
- $V_{n+1}(\mathbb{F}_q)$: $(n+1)$-dimensional vector space over $\mathbb{F}_q$;
- $G_k(V_{n+1})$: Grassmann geometry of the k-dimensional vector subspaces of V_{n+1};
- $\mathrm{PG}(V_{n+1}) = G_1(V_{n+1})$ *(column vectors)*;
- $\mathrm{PG}(V^*_{n+1}) = G_n(V_{n+1})$ *(row vectors)*.

Point–Hyperplane Geometry

$$\Gamma := (\mathscr{P}, \mathscr{L})$$

- $\mathscr{P}$: flags $([p],[\xi]) \subseteq \mathrm{PG}(V_{n+1}) \times \mathrm{PG}(V^*_{n+1})$ with $[p] \subseteq [\xi]$.
- $\mathscr{L}$: two types
  1. Given $\ell \in G_2(V_{n+1})$, $[\xi] \in \mathrm{PG}(V^*_{n+1})$, $\ell \subseteq [\xi]$:
  $$(\!(\ell,[\xi])\!) := \{([p],[\xi]) : p \in \ell\};$$
  2. Given $[p] \in \mathrm{PG}(V^*_{n+1})$, $S \in G_{n-1}(V_{n+1})$, $[p] \in S$:
  $$(\!([p],S)\!) := \{([p],[\xi]) : S \subseteq [\xi]\}.$$

Segre Embedding

Definition

- Segre Geometry: $\mathfrak{S}_{1,n} := \mathrm{PG}(V_{n+1}) \times \mathrm{PG}(V_{n+1}^*)$;
- Segre embedding: $\varepsilon : \mathfrak{S}_{1,n} \to \mathrm{PG}(V_{n+1} \otimes V_{n+1}^*)$;

$$\varepsilon(([p],[\xi])) = [p \otimes \xi] = [p \cdot \xi].$$

Remark

- $V_{n+1} \otimes V_{n+1}^* \cong M_{n+1}(\mathbb{F}_q)$;
- $\dim(\varepsilon) = \dim(V_{n+1} \otimes V_{n+1}^*)$;
- *Image of $\varepsilon(\mathfrak{S}_{1,n})$: projective points induced by all $(n+1)\times(n+1)$ matrices of rank 1.*

Segre Embedding

- $\Gamma \subseteq \mathfrak{S}_{1,n}$;
- $M^0_{n+1} := \{M \in M_{n+1} : \mathrm{Tr}(M) = 0\} \subseteq M_{n+1}(\mathbb{F}_q)$;
- $\varepsilon(\Gamma) \subseteq \mathrm{PG}(M^0_{n+1})$;
- $\varepsilon_1 := \varepsilon|_\Gamma$ is a projective embedding of Γ of dimension $n(n+2)$;
- The image Λ_1 of ε_1 consists of the projective points induced by all $(n+1)\times(n+1)$ matrices of rank 1 and trace 0.

Twisted embedding

- $\mathbb{F}_q$: field with q elements;
- $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, $\sigma \neq 1$: non-trivial automorphism.

Theorem

Let

$$\varepsilon_\sigma : \begin{cases} \Gamma \to \mathrm{PG}(V_{n+1} \otimes V_{n+1}^*) \cong \mathrm{PG}(M_{n+1}(q)) \\ ([p],[\xi]) \to [p^\sigma \otimes \xi] = [p^\sigma \cdot \xi]. \end{cases}$$

Then,

- ε_σ *is a projective embedding;*
- $\dim(\varepsilon_\sigma) = (n+1)^2$.

- $\Lambda_\sigma := \varepsilon_\sigma(\Gamma)$

Projective codes

- W: vector space over $\mathbb{F}_q$;
- $\dim(W) = k$;
- $\Omega \subseteq \mathrm{PG}(W)$: projective system;
- $\langle \Omega \rangle = \mathrm{PG}(W)$;
- $\mathscr{C}(\Omega)$: code with generator matrix whose columns correspond to the coordinates of the points of Ω.

Theorem (F.MacWilliams, 1964)

The code $\mathscr{C}(\Omega)$ has parameters $[N, d, k]$ where

$$N = |\Omega|, \qquad k = \dim(\langle \Omega \rangle)$$

$$d = N - \max_{H \in \mathrm{PG}(W*)} |\Omega \cap H|.$$

Minimal codes

- $\mathscr{C}(\Omega)$: code.
- $\mathrm{c} \in \mathscr{C}(\Omega)$.
- $\mathrm{supp}(\mathrm{c}) := \{i : c_i \neq 0\}$.

Definition

A codeword $\mathrm{c} \in \mathscr{C}(\Omega)$ is *minimal* if

$$\forall \mathrm{c}' \in \mathscr{C}(\Omega) : \mathrm{supp}(\mathrm{c}') \subseteq \mathrm{supp}(\mathrm{c}) \Rightarrow \exists \lambda \in \mathbb{F}_q : \mathrm{c}' = \lambda \mathrm{c}.$$

A code is *minimal* if all of its codewords are minimal.

Remark

Codewords in a minimal code are determined up to a non-zero scalar multiple by their support.

Parameters/Segre embedding ε_1

Theorem (I.Cardinali, LG 202?)

The code $\mathscr{C}_1 := \mathscr{C}(\Lambda_1)$ *is minimal and it has parameters* $[N_1, k_1, d_1]$ *given by*

$$N_1 = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}, \qquad k_1 = n^2+2n,$$

$$d_1 = q^{2n-1} - q^{n-1}.$$

Parameters/Twisted embedding ε_σ

Theorem (I.Cardinali, LG 202?)

If $\sigma \neq 1$, then the code $\mathscr{C}_\sigma := \mathscr{C}(\Lambda_\sigma)$ is minimal and it has parameters $[N_\sigma, k_\sigma, d_\sigma]$ given by

$$N_\sigma = \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}, \qquad k_\sigma = n^2+2n+1.$$

$$d_\sigma = \begin{cases} q^3 - \sqrt{q}^3 & \text{if } \sigma^2 = 1 \text{ and } n = 2, \\ q^{2n-1} - q^{n-1} & \text{if } \sigma^2 \neq 1 \text{ or } n > 2. \end{cases}$$

Weight spectrum/Segre embedding

Theorem (I.Cardinali, LG 202?)

1. *There is a bijection between*

$$\mathscr{I} := \{(g_1, \dots, g_t) \colon \sum_{i=1}^{t} g_i \leq n+1,\ 1 \leq g_1 \leq \dots \leq g_t \leq n+1$$

$$1 \leq t \leq q\} \cup \{0\}$$

   and the set of weights of $\mathscr{C}(\Lambda_1)$.
2. *The weights of* $\mathscr{C}(\Lambda_1)$ *are known.*
3. *It is possible to compute the weight enumerator.*

$\mathscr{C}(\Lambda_1)$: codewords

$$M \in M_{n+1}(\mathbb{F}_q)/\langle I \rangle, \qquad c_M := (\mathrm{Tr}(MX_1), \ldots, \mathrm{Tr}(MX_N)) \in \mathscr{C}(\Lambda_1)$$

Theorem (I. Cardinali, LG 202?)

- *The weight of a codeword c_M depends only on the number of eigenvectors of $M \in M_{n+1}(q)/\langle I \rangle$;*
- *The automorphism group of the code acts on the codewords as the product $\mathrm{PGL}(V_{n+1}) \cdot \mathbb{F}_q^*$ by the action*

$$([g], \alpha)(c_M) = c_{\alpha g^{-1} M g}.$$

$\mathscr{C}(\Lambda_1)$: codewords

Theorem (I.Cardinali, LG 202?)

- *Minimum weight codewords of* $\mathscr{C}(\Lambda_1)$ *are of the form* c_M *with* $\mathrm{rank}(M) = 1$ *and* $\mathrm{Tr}(M) \neq 0$

  $\varepsilon(([p],[\xi]))^{\perp}$ *with* $[p] \not\subseteq [\xi] \longleftrightarrow$ *points in* $\varepsilon(\mathfrak{S}_{1,n}) \setminus \Lambda_1$.
- *The minimum weight of* $\mathscr{C}(\Lambda_1)$ *is* $q^{2n-1} - q^{n-1}$.
- *The second lowest weight codewords are of the form* c_M *such that* $\mathrm{rank}(M) = 1$ *and* $\mathrm{Tr}(M) = 0$

  $\varepsilon(([p],[\xi]))^{\perp}$ *with* $[p] \subseteq [\xi] \longleftrightarrow$ *points in* Λ_1.
- *The second lowest weight of* $\mathscr{C}(\Lambda_1)$ *is* q^{2n-1}.
- *Maximum weight codewords are of the form* c_M *with* M *admitting no eigenvalue in* $\mathbb{F}_q$.
- *The maximum weight of* $\mathscr{C}(\Lambda_1)$ *is* $q^{n-1}(q^{n+1} - 1)/(q-1)$.

$\mathscr{C}(\Lambda_\sigma)$: codewords

$$M \in M_{n+1}(\mathbb{F}_q), \qquad c_M := (\mathrm{Tr}(MX_1), \dots, \mathrm{Tr}(MX_N)) \in \mathscr{C}(\Lambda_\sigma)$$

$$\theta_M := |\{\xi : [\xi]^\sigma \subseteq [\xi M]\}|$$

Theorem (I.Cardinali, LG 202?)

- *The weight of a codeword c_M depends only on θ_M.*
- *The group $\mathrm{GL}(V_{n+1})$ acts on the codewords as*

$$g(c_M) = c_{g^{-1}Mg^\sigma};$$

- *The full automorphism group of the code is isomorphic to* $\mathrm{PGL}(V_{n+1}) \cdot \mathbb{F}_q^*$.

$\mathscr{C}(\Lambda_\sigma)$: codewords

Theorem (I.Cardinali, LG 202?)

If $n > 2$ or $\sigma^2 \neq 1$, then

- *The minimum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight $q^{2n-1} - q^{n-1}$;*
- *The minimum weight codewords are of the form c_M where $M = \xi p^\sigma$ with $p\xi \neq 0$*
  $\varepsilon(([p],[\xi]))^\perp$ *with* $[p] \subseteq [\xi] \longleftrightarrow$ *points in* $\varepsilon(\mathfrak{S}_{1,n}) \setminus \Lambda_\sigma$.
- *The second lowest weight codewords have weight q^{2n-1};*
- *The second lowest weight codewords are of the form c_M where $M = \xi p^\sigma$ with $p\xi = 0$*
  $\varepsilon(([p],[\xi]))^\perp$ *with* $[p] \subseteq [\xi] \longleftrightarrow$ *points in* Λ_σ.
- *If both q and n are odd, then the maximum weight of $\mathscr{C}(\Lambda_\sigma)$ is $q^{n-1}(q^{n+1}-1)/(q-1)$.*

$\mathscr{C}(\Lambda_\sigma)$: codewords ($n=2, \sigma^2=1$)

Theorem (I.Cardinali, LG 202?)

- *If $n=2$ and $\sigma^2=1$, then the minimum weight codewords of $\mathscr{C}(\Lambda_\sigma)$ have weight $q^3-\sqrt{q}^3$ and are of the form c_M where M is such that there are three linearly independent row vectors ξ_1, ξ_2, ξ_3 and $\alpha, \beta, \gamma \in \mathbb{F}_q^*$ such that*

$$\alpha^{\sigma+1} = \beta^{\sigma+1} = \gamma^{\sigma+1}$$

$$\xi_1 M = \alpha \xi_1^\sigma, \quad \xi_2 M = \beta \xi_2^\sigma, \quad \xi_3 M = \gamma \xi_1^\sigma.$$

Small weight codewords

Theorem (I. Cardinali, LG 202?)

The codewords of minimum and second lowest weight of $\mathscr{C}(\Lambda_1)$ *and* $\mathscr{C}(\Lambda_\sigma)$ *are related to the* same *geometric hyperplanes of* Γ.

References

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Thank you for your attention