# Classification of low-degree ovoids

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## Definition

A **quadratic form** of $V$ is a map $\alpha : V \longrightarrow \mathbb{F}_q$ such that

- $\alpha(av) = \alpha^2 f(v)$
- $\beta(u, v) = \alpha(u+v) - \alpha(u) - \alpha(v)$ is a bilinear form

for any $u, v \in V$ and $a \in \mathbb{F}_q$.

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A **polarity** $\rho$ of $\mathrm{PG}(r,q)$ is a collineation between $\mathrm{PG}(r,q)$ and its dual space $\mathrm{PG}(r,q)^*$ such that $\rho^2 = \mathrm{id}$.

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An ovoid of $\mathcal{P}$ is a set of pairwise non-collinear points on $\mathcal{P}$

- $Q^{-}(2r+1,q)$ **elliptic quadric** of $\mathrm{PG}(2r+1,q)$

$$\alpha(x) = X_0X_{2r+1} + \ldots + X_rX_{r+3} + h(X_{r+1}, X_{r+2})$$

  $h$ is a homogeneous irreducible polynomial of degree 2 over $\mathbb{F}_q$
- $Q^{+}(2r+1,q)$ **hyperbolic quadric** of $\mathrm{PG}(2r+1,q)$

$$\alpha(x) = X_0X_{2r+1} + \ldots + X_rX_{r+3} + X_{r+1}X_{r+2}$$

- $Q(2r,q)$ **parabolic quadric** of $\mathrm{PG}(2r,q)$

$$\alpha(x) = X_0X_{2r} + \ldots + X_{r-1}X_{r+1} + X_r^2$$

- **symplectic polar space** of $\mathrm{PG}(2r+1,q)$

  $W(2r+1,q)$ full pointset of $\mathrm{PG}(2r+1,q)$
- **Hermitian variety** of $\mathrm{PG}(r,q^2)$

$$H(r,q^2) : X_0^{q+1} + \ldots + X_r^{q+1} = 0$$

$Q(4,q) : X_0X_4 + X_1X_3 + X_2^2 = 0$ parabolic quadric of $\mathrm{PG}(4,q)$

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Any ovoid of $Q(4,q)$ is equivalent to one of the form:

$$\mathcal{O}(f) = \{(1, x, y, f(x,y), -y^2 - xf(x,y))\}_{x,y \in \mathbb{F}_q} \cup \{(0,0,0,0,1)\}$$

for some polynomial $f$ s.t. $f(0,0) = 0$

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Known ovoids of $Q(4,q)$

| Name | $f(x,y)$ | Restrictions |
|---|---|---|
| *Elliptic quadric* | $-nx$ | $q$ odd, $n \notin \Box_q$ |
| *Elliptic quadric* | $ax + y$ | $q$ even, $a \neq 1, \mathrm{Tr}_{q/2}(a) = 1$ |
| *Kantor* | $-nx^\sigma$ | $q = p^h$ odd, $h > 1$<br>$n \notin \Box_q, \sigma \neq 1$ |
| *Penttila-Williams* | $-x^9 - y^{81}$ | $q = 3^5$ |
| *Thas-Payne* | $-nx - (n^{-1}x)^{1/9} - y^{1/3}$ | $q = 3^h, h > 2, n \notin \Box_q$ |
| *Ree-Tits slice* | $-x^{2\sigma+3} - y^\sigma$ | $q = 3^{2h+1}$, $\sigma = 3^{h+1}$ |
| *Tits* | $x^{\sigma+1} + y^\sigma$ | $q = 2^{2h+1}$, $\sigma = 2^{h+1}$ |

**Theorem (Bartoli, Durante - 2022)**

*If $q > 6.3(\deg(f) + 1)^{13/3}$ and $\mathcal{O}(f)$ is an ovoid of $Q(4, q)$, then $\mathcal{O}(f)$ is either an elliptic quadric or a Kantor ovoid.*

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$\beta$ symmetric form associated to the quadratic form of $Q(4,q)$

$$\mathcal{O}(f) \text{ is an ovoid} \iff \beta(P_1,P_2) \neq 0 \text{ for any } P_1 \neq P_2 \text{ in } \mathcal{O}(f) \iff$$
$$(y_1-y_2)^2 + (x_1-x_2)(f(x_2,y_2)-f(x_1,y_1)) \neq 0$$
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$\mathcal{S}_f$ hypersurface of $\mathrm{PG}(4,q)$ with equation

$$(X_2-X_4)^2X_0^{d-1} + (X_1-X_3)\Big(\tilde{f}(X_1,X_2,X_0) - \tilde{f}(X_4,X_3,X_0)\Big) = 0$$

$\widetilde{f}(x,y,t)$ homogenization of $f(x,y)$

$\mathcal{O}(f)$ is an ovoid $\iff$ $\mathcal{S}_f$ has no affine $\mathbb{F}_q$-rational points off the plane

$$X_1 - X_3 = X_2 - X_4 = 0$$

$\mathcal{O}(f)$ is an ovoid $\Longleftrightarrow$ $\mathcal{S}_f$ has no affine $\mathbb{F}_q$-rational points off the plane $X_1 - X_3 = X_2 - X_4 = 0$

**Theorem (Cafure, Matera - 2006)**

*Let $\mathcal{V} \subset \mathrm{AG}(n, q)$ be an absolutely irreducible $\mathbb{F}_q$-variety of dimension $r > 0$ and degree $\delta$. If $q > 2(r+1)\delta^2$, then the following estimate holds:*

$$|\#(\mathcal{V} \cap \mathbb{F}_q^n) - q^r| \leq (\delta - 1)(\delta - 2)q^{r-1/2} + 5\delta^{13/3}q^{r-1}.$$

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# Ovoids of $Q^+(5, q)$

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Any ovoid of $Q^+(5,q)$ is equivalent to one of the form:

$$\begin{aligned}\mathcal{O}(f_1,f_2) = &\{(0,0,0,0,0,1)\} \cup \\ &\{(1,x,y,f_1(x,y),f_2(x,y),-xf_2(x,y)-yf_1(x,y))\}_{x,y\in\mathbb{F}_q}\end{aligned}$$

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$\mathcal{O}(f_1,f_2)$ is an ovoid $\iff \beta(P_1,P_2)\neq 0$ for any $P_1\neq P_2$ in $\mathcal{O}(f_1,f_2) \iff$
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To a flock of the quadratic cone of $\mathrm{PG}(3, q)$ correponds a line spread of $\mathrm{PG}(3, q)$

We get classification results regarding ovoids associated to a flock of the quadratic cone of $\mathrm{PG}(3, q)$:

in this case $f_1(x, y) = y + g(x)$

Known ovoids of $Q^+(5,q)$ associated to flocks of the quadratic cone of $\mathrm{PG}(3,q)$

| Name | $f_1(x,y)$ | $f_2(x,y)$ | Restrictions |
|---|---|---|---|
| *Fisher-Thas-Walker* | $y-x^2$ | $x^3/3$ | $q \equiv -1 \pmod 3$ |
| *Kantor-Payne* | $y-\beta x^3$ | $\gamma x^5$ | $q=p^{2h+1}$<br>$p \equiv \pm 2 \pmod 5$<br>$\beta^2=5\gamma$ |
| *Law-Penttila* | $y-x^4-nx^2$ | $-n^{-1}x^9+x^7$<br>$+n^2x^3-n^3x$ | $q=3^h, n \notin \square_q$ |
| *Ganley* | $y-dx^3$ | $-n_1x^9-n_1n_2^2x$ | $q=3^h,\ h>2$<br>$n_1,n_2 \notin \square_q, d^2=n_1n_2$ |
| *Kantor* | $y-x^2$ | $\frac{1}{3}x^3-nx^5-n^{-1}x$ | $q=5^h, n \notin \square_q$ |
| *Penttila-Williams* | $y+x^{27}$ | $-x^9$ | $q=3^5$ |

**Theorem (Bartoli, Durante, G. - 2024)**

*If $q > 6.3(\max\{\deg(f_1),\deg(f_2)\}+1)^{13/3}$ and $\mathcal{O}(f_1,f_2)$ is an ovoid of $Q^+(5,q)$, then $\mathcal{O}(f_1,f_2)$ is either a Fisher-Thas-Walker ovoid or a Kantor-Payne ovoid or a Law-Penttila ovoid or a Ganley ovoid or a Kantor ovoid.*

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$$\mathcal{O}(f_1, f_2) = \{(0, 0, 0, 0, 0, 0, 1)\} \cup$$
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for any $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ in $\mathbb{F}_q^3$

$\mathcal{W}_{f_1,f_2}$ hypersurface of $\mathrm{PG}(6,q)$ with equation

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$\widetilde{f_i}(x,y,z,t)$ homogenization of $f_i(x,y,z)$, $i\in\{1,2\}$

$\mathcal{O}(f_1,f_2)$ is an ovoid $\iff \mathcal{W}_{f_1,f_2}$ has no affine $\mathbb{F}_q$-rational points off the solid $X_1-X_4=X_2-X_5=X_3-X_6=0$

**Theorem (Bartoli, Durante, G. - 2024)**

*If $q>6.3(\max\{\deg(f_1),\deg(f_2)\}+1)^{13/3}$ and $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component $\mathcal{V}$ over $\mathbb{F}_q$, then $\mathcal{O}(f_1,f_2)$ is not an ovoid of $Q(6,q)$.*

Known ovoids of $Q(6, q)$

| Name | $f_1(x, y, z)$ | $f_2(x, y, z)$ | Restrictions |
|---|---|---|---|
| Thas-Kantor | $-ny^3 + x^2y - xz$ | $-1/nx^3 + xy^2 + yz$ | $q = 3^h$, $n \notin \square_q$ |
| Ree-Tits | $-x^{\sigma+3} + y^\sigma$ $+x^2y - xz$ | $-x^{2\sigma+3} + x^\sigma y^\sigma - z^\sigma$ $+xy^2 + yz$ | $q = 3^{2h+1}$, $\sigma = 3^{h+1}$ |

**Theorem (Bartoli, Durante, G. - 2024)**

*If $q > 6.3(\max\{\deg(f_1), \deg(f_2)\} + 1)^{13/3}$ and $\mathcal{O}(f_1, f_2)$ is an ovoid of $Q(6, q)$, then $\mathcal{O}(f_1, f_2)$ is a Thas-Kantor ovoid.*

# Ovoids of $Q^+(7, q)$

$Q^+(7, q) : X_0X_7 + X_1X_6 + X_2X_5 + X_3X_4 = 0$ hyperbolic quadric of $\mathrm{PG}(7, q)$

# Ovoids of $Q^+(7, q)$

$Q^+(7, q) : X_0X_7 + X_1X_6 + X_2X_5 + X_3X_4 = 0$ hyperbolic quadric of $\mathrm{PG}(7, q)$

Any ovoid of $Q^+(7, q)$ is equivalent to one of the form:

$$\mathcal{O}(f_1, f_2, f_3) = \{P_{x,y,z}\}_{x,y,z\in\mathbb{F}_q} \cup \{(0, 0, 0, 0, 0, 0, 0, 1)\}$$

$P_{x,y,z} =$
$(1, x, y, z, f_1(x, y, z), f_2(x, y, z), f_3(x, y, z), -zf_1(x, y, z) - yf_2(x, y, z) - xf_3(x, y, z))$

for some polynomials $f_1, f_2, f_3$ s.t. $f_i(0, 0, 0) = 0$, $i = 1, 2, 3$

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$$(1,x,y,z,f_1(x,y,z),f_2(x,y,z),f_3(x,y,z),-zf_1(x,y,z)-yf_2(x,y,z)-xf_3(x,y,z))$$

for some polynomials $f_1,f_2,f_3$ s.t. $f_i(0,0,0)=0$, $i=1,2,3$

$\beta$ symmetric form associated to the quadratic form of $Q^+(7,q)$

$\mathcal{O}(f_1,f_2,f_3)$ is an ovoid $\iff \beta(P_1,P_2)\neq 0$ for any $P_1\neq P_2$ in $\mathcal{O}(f_1,f_2,f_3)$

$$\begin{gathered}\iff (x_1-x_2)(f_3(x_2,y_2,z_2)-f_3(x_1,y_1,z_1))\\ +(y_1-y_2)(f_2(x_2,y_2,z_2)-f_2(x_1,y_1,z_1))\\ +(z_1-z_2)(f_1(x_2,y_2,z_2)-f_1(x_1,y_1,z_1))\neq 0\end{gathered}$$

for any $(x_1,y_1,z_1)\neq(x_2,y_2,z_2)$ in $\mathbb{F}_q^3$

$\mathcal{S}_{f_1,f_2,f_3}$ hypersurface of $\mathrm{PG}(6,q)$ with equation

$$\begin{aligned}
&(X_1 - X_4)\Big(\widetilde{f_3}(X_4,X_5,X_6,X_0) - \widetilde{f_3}(X_1,X_2,X_3,X_0)\Big)+\\
&(X_2 - X_5)\Big(\widetilde{f_2}(X_4,X_5,X_6,X_0) - \widetilde{f_2}(X_1,X_2,X_3,X_0)\Big)+\\
&(X_3 - X_6)\Big(\widetilde{f_1}(X_4,X_5,X_6,X_0) - \widetilde{f_1}(X_1,X_2,X_3,X_0)\Big) = 0
\end{aligned}$$

$\widetilde{f_i}(x,y,z,t)$ homogenization of $f_i(x,y,z)$, $i \in \{1,2\}$

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$$(X_3 - X_6)\Big(\widetilde{f_1}(X_4,X_5,X_6,X_0) - \widetilde{f_1}(X_1,X_2,X_3,X_0)\Big) = 0$$

$\widetilde{f_i}(x,y,z,t)$ homogenization of $f_i(x,y,z)$, $i \in \{1,2\}$

$\mathcal{O}(f_1,f_2,f_3)$ is an ovoid $\Longleftrightarrow$ $\mathcal{S}_{f_1,f_2,f_3}$ has no affine $\mathbb{F}_q$-rational points off the solid $X_1 - X_4 = X_2 - X_5 = X_3 - X_6 = 0$

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*If $q > 6.3(\max\{\deg(f_1),\deg(f_2),\deg(f_3)\}+1)^{13/3}$ and $\mathcal{S}_{f_1,f_2,f_3}$ contains an absolutely irreducible component $\mathcal{V}$ over $\mathbb{F}_q$, then $\mathcal{O}(f_1,f_2,f_3)$ is not an ovoid of $Q^+(7,q)$.*

Known ovoids of $Q^{+}(7, q)$

| Name | Restrictions |
|---|---|
| *Thas-Kantor* | $q = 3^h,\ h > 0$ |
| *Ree-Tits* | $q = 3^{2h+1},\ h > 0$ |
| *Kantor* (1) | $q = p^h,\ p \equiv 2 \pmod 3$ (prime), $h$ odd |
| *Kantor* (2) | $q = 2^h,\ h \geq 1$ |
| *Dye* | $q = 8$ |

Thas-Kantor and Ree-Tits are ovoids of $Q(6, q)$

Dye ovoid:

$$f_1(x, y, z) = x + y + z + x^2y + x^4y^2 + xy^2 + x^2y^4 + x^4y^4$$
$$f_2(x, y, z) = y + x^2z + x^4z^2 + xz^2 + x^2z^4 + x^4z^4$$
$$f_3(x, y, z) = x + y + y^2z + y^4z^2 + yz^2 + y^2z^4 + y^4z^4$$

Kantor (1) ovoid, $q \equiv 2 \pmod 3$

$$V = \left\{ M = \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \beta^q \\ b & \gamma^q & \alpha^q \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{F}_{q^2}, a, b, c \in \mathbb{F}_q, a + \alpha + \alpha^q = 0 \right\}$$

$$Q(M) = \mathrm{Tr}_{q^2/q}(\alpha)^2 - \mathrm{N}_{q^2/q}(\alpha) + \mathrm{Tr}_{q^2/q}(\beta\gamma) + bc$$

$$\mathcal{O} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \mu^q & \mu^q\rho & \mu^{q+1} \\ \rho^q & \rho^{q+1} & \mu\rho^q \\ 1 & \rho & \mu \end{pmatrix} : \mathrm{Tr}_{q^2/q}(\mu) + \mathrm{N}_{q^2/2}(\rho) = 0 \right\}$$

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*The set $\mathcal{O}$ corresponds to the ovoid $\mathcal{O}(f_1, f_2, f_3)$, where if $q$ is odd*

$$\begin{aligned} f_1(x,y,z) &= -6xy - 3y^3 - 9z^3 - z^2y - 3y^2z \\ f_2(x,y,z) &= -y^3 - 3yz^2 + 6xz + 3y^2z + 9z^3 \\ f_3(x,y,z) &= -3x - 3y^2 - 9z^2, \end{aligned}$$

*if $q$ is even*

$$\begin{aligned} f_1(x,y,z) &= z^3 + z^2y + zy^2 + xy \\ f_2(x,y,z) &= z^3 + y^3 + xz \\ f_3(x,y,z) &= x + z^2 + zy + y^2. \end{aligned}$$

Kantor (2) ovoid, $q = 2^h, h \geq 1$

$$V = \mathbb{F}_q \oplus \mathbb{F}_{q^3} \oplus \mathbb{F}_{q^3} \oplus \mathbb{F}_q$$

$$Q((a,\gamma,\delta,d)) = ad + \mathrm{Tr}_{q^3/q}(\gamma\delta)$$

$$\mathcal{O} = \{(0,0,0,1)\} \cup \{(1,t,t^{q+q^2},\mathrm{N}_{q^3/q}(t)) : t \in \mathbb{F}_{q^3}\}$$

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let $\{1,\alpha,\beta\}$ be an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^3}$, and put $t = x + y\alpha + z\beta$ with $x,y,z \in \mathbb{F}_q$. Then $\mathcal{O}$ corresponds to the ovoid $\mathcal{O}(f_1,f_2,f_3)$, where*

$$f_1(x,y,z) = \mathrm{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})xy + \mathrm{Tr}_{q^3/q}(\beta)x^2 + \mathrm{Tr}_{q^3/q}(\alpha^{q+1}\beta^{q^2})y^2 + \mathrm{N}_{q^3/q}(\beta)z^2$$

$$f_2(x,y,z) = \mathrm{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})xz + \mathrm{Tr}_{q^3/q}(\alpha)x^2 + \mathrm{N}_{q^3/q}(\alpha\beta^{q^2})y^2 + \mathrm{Tr}_{q^3/q}(\alpha\beta^{q^2+q})z^2$$

$$f_3(x,y,z) = \mathrm{Tr}_{q^3/q}(\alpha\beta^2 + \alpha\beta^{q^2})yz + x^2 + \mathrm{Tr}_{q^3/q}(\alpha^{q+1}\beta^{q^2})y^2 + \mathrm{Tr}_{q^3/q}(\beta^{q+1})z^2.$$

## CASE $d = 2$

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let $q > 6.3 \cdot 3^{13/3}$. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $f_1, f_2, f_3$ have degree 2, then $q$ is even and $\mathcal{O}(f_1, f_2, f_3)$ is the Kantor (2) ovoid.*

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CASE $d = 3$

If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $q > 6.3 \cdot 4^{13/3}$, then $\mathcal{S}_{f_1,f_2,f_3}$ does not contain any absolutely irreducible component defined over $\mathbb{F}_q$

Since $\mathcal{S}_{f_1,f_2,f_3}$ is of degree 4, it must split into either four hyperplanes or two quadrics

## CASE $d = 2$

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let $q > 6.3 \cdot 3^{13/3}$. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $f_1, f_2, f_3$ have degree 2, then $q$ is even and $\mathcal{O}(f_1, f_2, f_3)$ is the Kantor (2) ovoid.*

## CASE $d = 3$

If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$ and $q > 6.3 \cdot 4^{13/3}$, then $\mathcal{S}_{f_1,f_2,f_3}$ does not contain any absolutely irreducible component defined over $\mathbb{F}_q$

Since $\mathcal{S}_{f_1,f_2,f_3}$ is of degree 4, it must split into either four hyperplanes or two quadrics

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Assume $q > 6.3 \cdot 4^{13/3}$ and $f_1, f_2, f_3$ of degree 3. If $\mathcal{S}_{f_1,f_2,f_3}$ splits into four hyperplanes, then the set $\mathcal{O}(f_1, f_2, f_3)$ is not an ovoid of $Q^+(7, q)$.*

$\mathcal{S}_{f_1,f_2,f_3}$ splits into two quadrics
$p > 3$ and $p = 2$

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**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let* $q > 6.3 \cdot 4^{13/3}$, $q \equiv 1 \pmod 3$, $p > 3$, $f_1, f_2, f_3$ *of degree* 3, *and* $\mathcal{S}_{f_1,f_2,f_3}$ *split into two quadrics. The set* $\mathcal{O}(f_1, f_2, f_3)$ *is not an ovoid of* $Q^+(7, q)$.

$\mathcal{S}_{f_1,f_2,f_3}$ splits into two quadrics
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**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let $q > 6.3 \cdot 4^{13/3}$, $q \equiv 1 \pmod 3$, $p > 3$, $f_1, f_2, f_3$ of degree 3, and $\mathcal{S}_{f_1,f_2,f_3}$ split into two quadrics. The set $\mathcal{O}(f_1, f_2, f_3)$ is not an ovoid of $Q^+(7, q)$.*

**Theorem (Bartoli, Durante, G., Timpanella - 202x)**

*Let $q > 6.3 \cdot 4^{13/3}$, $q \equiv 2 \pmod 3$, $p = 2$, and $f_1, f_2, f_3$ of degree 3. Assume that $\mathcal{S}_{f_1,f_2,f_3}$ splits into two quadrics. If $\mathcal{O}(f_1, f_2, f_3)$ is an ovoid of $Q^+(7, q)$, then*

$$\begin{aligned}
f_1(X,Y,Z) &= Z^3 + Y^2Z + YZ^2 + (A+B)Y^2 + BZ^2 + XY \\
&\quad + CX + DY + B^2Z, \\
f_2(X,Y,Z) &= Y^3 + Z^3 + AY^2 + (A+B)Z^2 + XZ + EX + A^2Y + DZ, \\
f_3(X,Y,Z) &= Y^2 + Z^2 + YZ + X + EY + CZ.
\end{aligned}$$

## Theorem (Bartoli, Durante, G., Timpanella - 202x)

*Let* $q > 6.3 \cdot 4^{13/3}$*,* $q \equiv 2 \pmod 3$*,* $p > 3$*, and* $f_1, f_2, f_3$ *of degree* 3*. Assume that* $\mathcal{S}_{f_1,f_2,f_3}$ *splits into two quadrics. If* $\mathcal{O}(f_1, f_2, f_3)$ *is an ovoid of* $Q^+(7, q)$*, then*

$$\begin{aligned} f_1(X,Y,Z) &= -\frac{4}{27}Z^3 + \frac{4\epsilon}{3}Y^3 - \frac{4}{9}Y^2Z + \frac{4\epsilon}{9}YZ^2 + \left(\frac{4\epsilon}{3}A - 2B\right)Y^2 - \frac{2}{3}BZ^2 + \\ &\quad + \frac{4\epsilon}{3}XY + \frac{4\epsilon}{3}BZ + CY - B^2Z, \\ f_2(X,Y,Z) &= -\frac{4}{3}Y^3 - \frac{4\epsilon}{9}Z^3 - \frac{4}{9}YZ^2 + \frac{4\epsilon}{3}Y^2Z - 2C_4Y^2 - \left(\frac{4\epsilon}{3}A - \frac{2}{3}B\right)Z^2 \\ &\quad - \frac{4\epsilon}{3}XZ + \frac{4\epsilon}{3}AYZ + DX - A^2Y - (2AB + C)Z, \\ f_3(X,Y,Z) &= -2Y^2 - \frac{2}{3}Z^2 - X - (2A + D)Y - (2B + E)Z, \end{aligned}$$

*where* $\epsilon = \pm 1$*.*

**Open Problem**

*Determine whether for $f_1, f_2, f_3$ defined as in the previous statements, the ovoid $\mathcal{O}(f_1, f_2, f_3)$ is equivalent to the Kantor $(1)$ ovoid.*

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## Open Problem

*Obtain a full classification for ovoids $\mathcal{O}(f_1, f_2, f_3)$ such that $f_1, f_2, f_3$ have degree $3$ when $p = 3$.*

*Thanks for your attention!*