Erdős-Ko-Rado problems and Uniqueness

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joint work with
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Finite Geometries 2025

The EKR problem

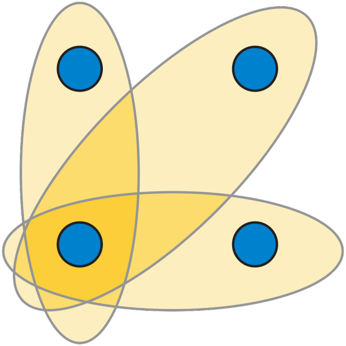

Figure: Star-shaped EKR-set [1]

[1] https://upload.wikimedia.org/wikipedia/commons/8/86/Intersecting_set_families_2-of-4.svg

Kneser graphs

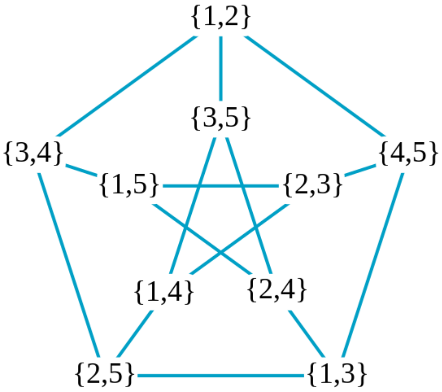

Figure: The Kneser graph $K(5,2)$ [2]

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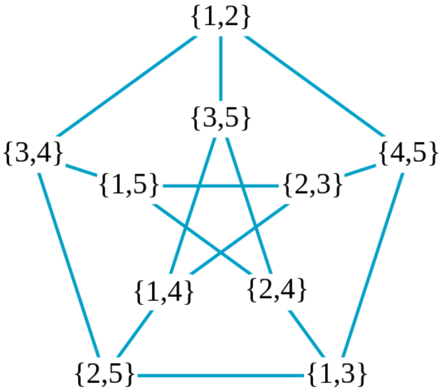

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EKR-sets are cocliques of the Kneser graph.

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Chambers of projective planes

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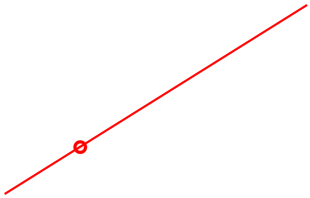

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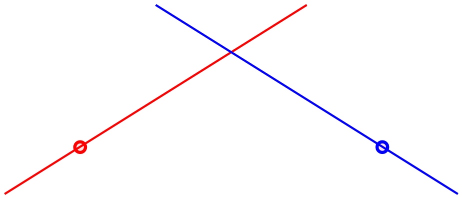

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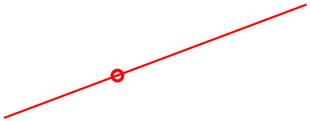

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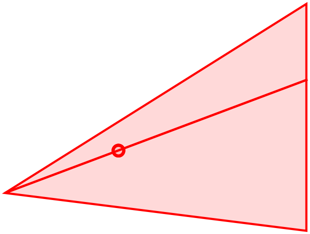

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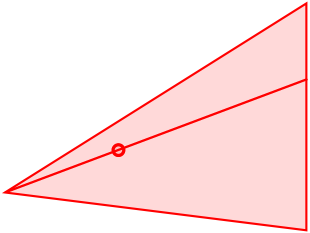

A chamber C is a tuple $C = (C_0, \dots, C_{n-1})$.

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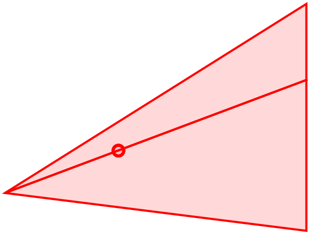

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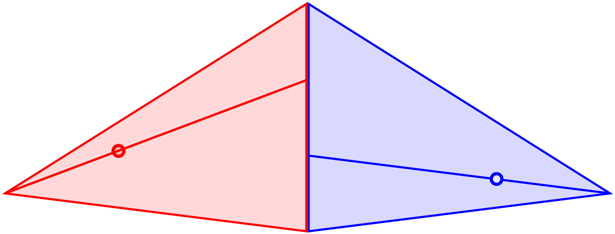

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EKR problem for chambers of projective spaces

Let $\mathcal{F}$ be a set of pairwise non-opposite chambers of $\mathrm{PG}(n, q)$.

How big can $\mathcal{F}$ be?
What is the structure of $\mathcal{F}$?

The Hoffman ratio-bound

Let $\Gamma = (X, E)$ be a regular graph of degree d and smallest eigenvalue λ_{min}.

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Theorem (Hoffman ratio-bound)

$$\alpha(\Gamma) \leq |X| \frac{-\lambda_{min}}{d - \lambda_{min}}$$

EKR-size for chambers of $\mathbb{F}_q^n$

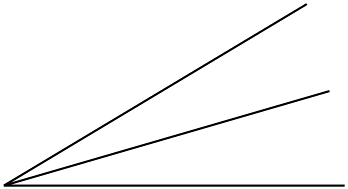
Theorem [De Beule, Mattheus, Metsch 2022]

Let $\mathcal{F}$ be a set of pairwise non-opposite chambers of $\mathrm{PG}(n,q)$. Then

$$|\mathcal{F}| \leq \frac{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}^2 \cdot \ldots \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{1+q^{(n+1)/2}}.$$

Classical examples that meet the bound

for n odd



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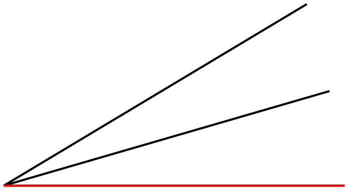

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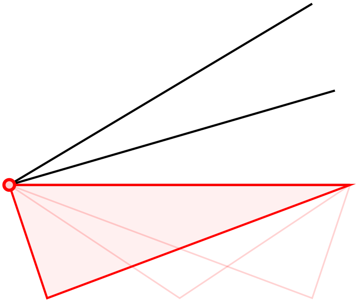

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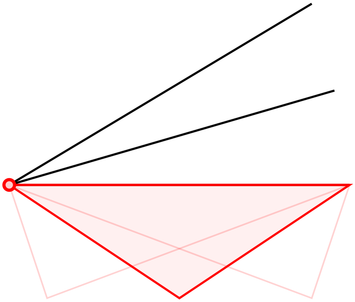

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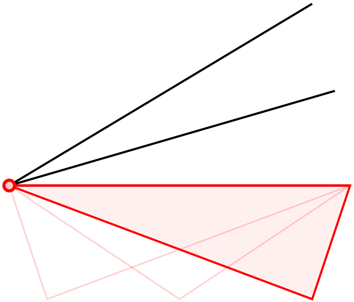

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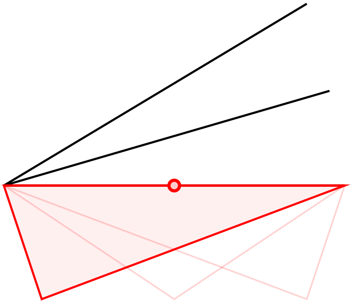

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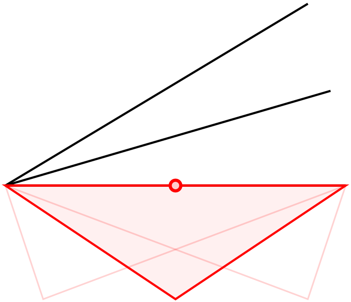

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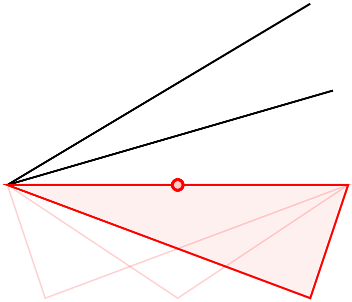

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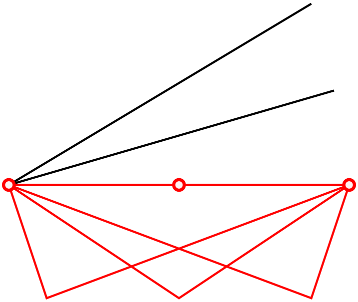

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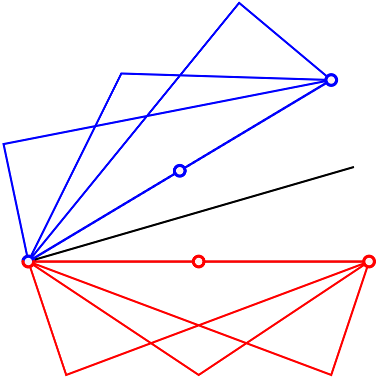

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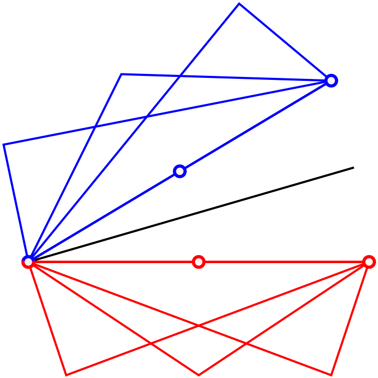

Pairwise intersecting $(n+1)/2$-subspaces
$\rightarrow$ Pairwise non-opposite chambers

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Theorem [H., Lansdown, Metsch 2025]

Consider a projective space $\mathrm{PG}(n, q)$ with n odd.

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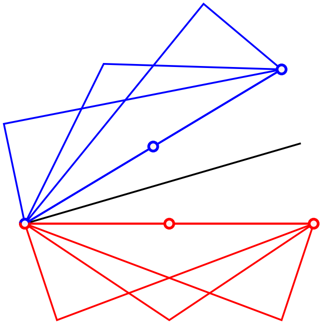

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An *antidesign* is a vector w such that $v^\top w = 0$ for all $v \in V_{min}$.

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If w is an antidesign, then

$$\mathbb{1}_{\mathcal{F}}^{\top} w = \frac{\mathbb{1}^{\top} w}{q^{(n+1)/2} + 1}$$

How to get Antidesigns?

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For a chamber $C = (C_0, \ldots, C_{n-1})$ this means

$$w_C(B) := \begin{cases} -\lambda_{min} & \text{if } C = B, \\ 1 & \text{if } C \text{ and } B \text{ are opposite,} \\ 0 & \text{otherwise.} \end{cases}$$

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Let S be a spread of $(n+1)/2$-spaces.

$$w_S(B) = \begin{cases} 1 & \text{if } B_{(n+1)/2} \in S, \\ 0 & \text{otherwise.} \end{cases}$$

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Thank you for your attention