New Distance-Biregular Graphs

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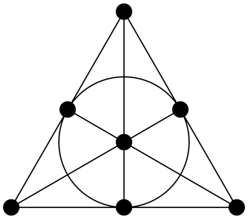
Finite Geometries, Irsee 2025

The Incidence Graph of the Fano Plane ($PG(2,2)$)

Vertices: Points and Lines.
Adjacency: Incidence.

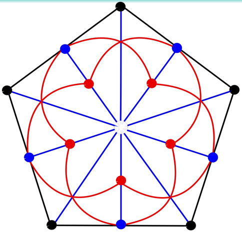


This is a **distance-regular graph**:

Take two vertices x, y at distance k. Then the number of vertices z at distance i from x and j from y only depends on i, j, k.

Example: Take a non-incident point-line pair (P, L). Then there are precisely 3 lines through P which meet L.

The Incidence Graph of the Doily ($GQ(2,2)$)



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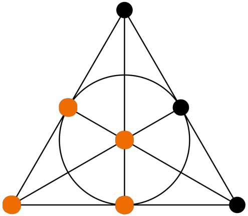
Take two vertices x, y at distance k. Then the number of vertices z at distance i from x and j from y only depends on i, j, k.

Example: Take a non-incident point-line pair (P, L). Then there is precisely 1 line through P which meets L.

The Incidence Graph of a $GQ(s,t)$

Take a **hyperoval** $\mathcal{O}$ of a projective plane $\pi \cong PG(2,q)$.

Vertices Y: Points of $PG(3,q) \setminus \pi$.
Vertices Z: Lines of $PG(3,q) \setminus \pi$ meeting $\mathcal{O}$.
Adjacency: Incidence.



This is a **not a distance-regular graph**.
It is a generalized quadrangle of order $(q-1, q+1)$.

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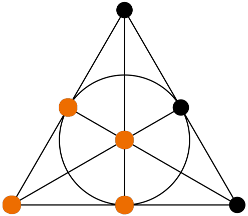

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It is a **distance-biregular graph** (DBRG)!

We have a bipartite graph with parts Y and Z (points and lines). Take two vertices x, y at distance k. Then the number of vertices z at distance i from x and j from y only depends on i, j, k and on whether $x \in Y$ or $x \in Z$.

Is the concept of DBRGs interesting?
Are there objects which are DBRG, but nothing else?

Delorme's Construction from an Hyperbolic Quadric

Consider the **Klein quadric** $\mathcal{Q} \cong Q^+(5,q)$ in $H \cong PG(5,q)$.
The Klein quadric has $2(q^3+q^2+q+1)$ **planes**: half Latins, half Greeks.

Vertices Y**:** Points of $PG(6,q) \setminus H$.
Vertices Z**:** Solids of $PG(6,q)$ meeting H in a Greek of $\mathcal{Q}$.
Adjacency: Incidence.

DBRG with parts of sizes q^6 and $(q+1)(q^2+1) \cdot q^3$.

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Till recently, **no new examples**:

- Delorme (1984/1994): from maximal arcs.
- Van Den Akker (1990): Hall-Janko-Wales graph.
- Fernández, Ih., Lato, Munemasa (2025*):
  - One new sporadic example.
  - One new infinite family.

A kind of Two-Intersection Sets

Consider $V = V(n, q)$. Let $\mathcal{S}^*$ be a family of s subspaces of $V(n, q)$ of co-dimension k in V such that

1. each $v \in V^*$ lies in 0 or d elements of $\mathcal{S}^*$,
2. we have $\dim(M \cap M^*) = n - 2k$ for all $M, M^* \in \mathcal{S}^*$.

Question: Do you know examples?

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Question: Do you know examples?

1. Maximal arcs in $PG(2, 2^h)$.
2. Blow-ups of maximal arcs to $PG(3\ell - 1, 2^{h/\ell})$.
3. A construction by Mathon (2002) with $(n,k,q,d,s) = (6,2,3,3,21)$.

Mathon's Construction is fascinating, **ask Simeon Ball about it!**

DBRGs from Two-Intersection Sets

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1. each $v \in V^*$ lies in 0 or d elements of $\mathcal{S}^*$,
2. we have $\dim(M \cap M^*) = n - 2k$ for all $M, M^* \in \mathcal{S}^*$.

Identify V with $H \cong PG(n-1,q)$ at infinity of $AG(n,q)$.

Theorem (Fernández, Ih., Lato, Munemasa (2025*))

Let Y be the points of $AG(n,q)$ and Z be the $(k+1)$-spaces of $AG(n,q)$ meeting H in an elements $\mathcal{S}^*$. Then the bipartite incidence graph on $Y \cup Z$ is distance-biregular with intersection array

$$\left|\begin{matrix} s; & 1, & d, & q^{n-2k}(s-1)/d, & s \\ q^{n-k}; & 1, & q^{n-2k}, & s-1, & q^{n-k} \end{matrix}\right|.$$

The only new example from this is **Mathon's Construction**.

Dual Hyperovals

Take a **dual hyperoval** $\mathcal{O}^*$ of a projective plane $\pi \cong PG(2,q)$.

Vertices Y: Points of $PG(3,q) \setminus \pi$.
Vertices Z: Planes of $PG(3,q) \setminus \pi$ meeting π in an element of $\mathcal{O}^*$.
Adjacency: Incidence.

This gives a **distance-biregular graph** (earlier slide).

Long known: Delorme (1984/1994).

Derived Hyperovals

Take a **dual hyperoval** $\mathcal{O}^*$ of a projective plane $\pi \cong PG(2,q)$.
Fix a **point** P of $PG(3,q) \setminus \pi$.

We look at the distance-3-or-4-neighborhoods of P in our previous graph!

Derived Hyperovals

Take a **dual hyperoval** $\mathcal{O}^*$ of a projective plane $\pi \cong PG(2,q)$.
Fix a **point** P of $PG(3,q) \setminus \pi$.

We look at the distance-3-or-4-neighborhoods of P in our previous graph!

Vertices Y: Only points Q s.t. $P+Q$ meets π in **exterior point**.
Vertices Z: Only planes not containing P.
Adjacency: Incidence.

Proof 1: Geometry and counting.
Proof 2:

Theorem (Fernández, Ih., Lato, Munemasa (2025*))

If the parameters of a distance-biregular graph of diameter 4 satisfy certain conditions, then $N_3(z) \cup N_4(z)$ is distance-biregular too.

Application 1: **hyperovals**.
Application 2: **nonexistence**.

As a Picture

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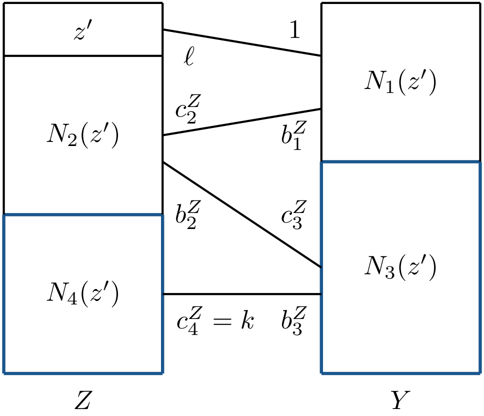

Open Problem

Just look at our tables and do numerology:

Intersection Array	Halved Graph	Notes
$\begin{Bmatrix} 6; & 1, & 2, & 10, & 6 \\ 16; & 1, & 4, & 5, & 16 \end{Bmatrix}$	$(64, 45, 32, 30)$ $(24, 20, 16, 20)$	Delorme [21] Ex. 3.3.2: $q = 4, r = 2$
$\begin{Bmatrix} 8; & 1, & 2, & 6, & 8 \\ 15; & 1, & 3, & 4, & 15 \end{Bmatrix}$	$(120, 56, 28, 24)$ $(64, 35, 18, 20)$	Delorme [21] Ex. 3.3.1: $q = 2$
$\begin{Bmatrix} 10; & 1, & 2, & 18, & 10 \\ 28; & 1, & 4, & 9, & 28 \end{Bmatrix}$	$(196, 135, 94, 90)$ $(70, 63, 56, 63)$	Constr. 6.2.2 $q = 2$
$\begin{Bmatrix} 8; & 1, & 2, & 21, & 8 \\ 36; & 1, & 6, & 7, & 36 \end{Bmatrix}$	$(216, 140, 94, 84)$ $(48, 42, 36, 42)$	Van Den Akker [1] Section 6.2
$\begin{Bmatrix} 15; & 1, & 3, & 28, & 15 \\ 36; & 1, & 6, & 14, & 36 \end{Bmatrix}$	$(216, 175, 142, 140)$ $(90, 84, 78, 84)$	Only known SRG [16] does not work
$\begin{Bmatrix} 12; & 1, & 3, & 33, & 12 \\ 45; & 1, & 9, & 11, & 45 \end{Bmatrix}$	$(225, 176, 139, 132)$ $(60, 55, 50, 55)$	Corollary 4.2.3 $\gamma_2 = \frac{9}{5}$
$\begin{Bmatrix} 10; & 1, & 2, & 12, & 10 \\ 28; & 1, & 4, & 6, & 28 \end{Bmatrix}$	$(280, 135, 70, 60)$ $(100, 63, 38, 42)$	Van Den Akker [1] Ex. 3.3.3
$\begin{Bmatrix} 15; & 1, & 3, & 20, & 15 \\ 36; & 1, & 6, & 10, & 36 \end{Bmatrix}$	$(288, 175, 110, 100)$ $(120, 84, 58, 60)$	

Thank you for your attention!