About Reed-Muller codes $RM_q(2,2)$

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There are several ways of producing linear codes from geometry or algebra.

Geometry and Coding Theory: A Selective Dictionary

Veronesean $\mathcal{V}_{r,m}$:
image of the r-uple embedding
$$\mathbb{P}^m(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{\binom{m+r}{r}-1}(\mathbb{F}_q)$$

Projective Reed-Muller code $\mathrm{PRM}_q(r,m)$:
Evaluation of $\mathbb{F}_q[x_0, x_1, \ldots, x_m]_r$ on
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Hyperplane sections
$$|H \cap \mathcal{V}_{r,m}|; \quad |H \cap \mathcal{V}_{r,m}^{\mathbb{A}}|$$
$$\max_H |H \cap \mathcal{V}_{r,m}|; \quad \max_H |H \cap \mathcal{V}_{r,m}^{\mathbb{A}}|$$

Weight distribution of
$\mathrm{PRM}_q(r,m)$; $\mathrm{RM}_q(r,m)$

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Minimum distance of $\mathrm{PRM}_q(r,m)$; $\mathrm{RM}_q(r,m)$

Linear sections
$$|L \cap \mathcal{V}_{r,m}|; \quad |L \cap \mathcal{V}^{\mathbb{A}}_{r,m}|$$

Higher weight spectra of
$\mathrm{PRM}_q(r,m)$; $\mathrm{RM}_q(r,m)$

$$\max\{|L \cap \mathcal{V}_{r,m}| : \operatorname{codim} L = i\};$$
$$\max\{|L \cap \mathcal{V}^{\mathbb{A}}_{r,m}| : \operatorname{codim} L = i\}$$

i-th Generalized Hamming weight of
$\mathrm{PRM}_q(r,m)$; $\mathrm{RM}_q(r,m)$

Higher Weight Spectra of a linear $[n, k]_q$-code C

- The support and the support weight of a subcode D of C:

$$\mathrm{Supp}(D) := \{i : \exists\ c = (c_1, \ldots, c_n) \in D \text{ with } c_i \neq 0\},\ \ \mathsf{wt}(D) := |\mathrm{Supp}(D)|.$$

- The i-th generalized Hamming weight (GHW) of C $(1 \leq i \leq k)$:

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- For $0 \leq i \leq k$, the i-th weight spectrum of C is the ordered multiset $\{A_w^{(i)}(C)\}$ for $w = 0, \ldots, n$, where

$$A_w^{(i)}(C) = |\{D : D \subseteq C, \dim D = i, \text{and } \mathsf{wt}(D) = w\}|$$

and we call the multiset of i-th weight spectra for $i = 0, 1, \ldots, k$ as the higher weight spectra of C.

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  and we call the multiset of i-th weight spectra for $i = 0,1,\ldots,k$ as the higher weight spectra of C. In particular, $A_0(C) = 1 = A_0^{(0)}(C)$ and $A_w(C) = (q-1)A_w^{(1)}(C)$ for $1 \le w \le n$. We will write $A_w^{(i)}$ for $A_w^{(i)}(C)$ whenever the code C is understood from the context.

What is Known

- [Kasami–Lin–Peterson (1968)] The minimum distance of $\mathrm{RM}_q(r, m)$
- [McEliece (1969) $+$ Li (2019)] The weight distribution of the second order affine and projective Reed-Muller codes $\mathrm{RM}_q(2, m)$ and $\mathrm{PRM}_q(2, m)$.
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Question: What about the complete higher weight spectra of $\mathrm{RM}_q(2,2)$?

What do we know about $\mathrm{RM}_q(2,2)$?

$\mathrm{RM}_q(2,2)$ is a $[q^2, 6, q^2 - 2q]_q$-code if $q > 2$, and is a $[4,3,2]_2$-code if $q = 2$. Thanks to various prior works, we already know the following for any $q > 2$:

- The nonzero weights of $\mathrm{RM}_q(2,2)$ are

$$q^2 - 2q, \quad q^2 - 2q + 1, \quad q^2 - q - 1, \quad q^2 - q + 1$$

- The generalized Hamming weights of $\mathrm{RM}_q(2,2)$ are

$$d_1 = q^2 - 2q, \quad d_2 = q^2 - 4, \quad d_3 = q^2 - 3, \quad d_4 = q^2 - 2, \quad d_5 = q^2 - 1, \quad d_6 = q^2.$$

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Main Result: Explicit formulas for the higher weight spectra of $\mathrm{RM}_q(2,2)$.

Higher weight spectra of $\mathrm{RM}_q(2,2)$ when $q \geq 7$

$A_0^{(0)} = 1, \quad A_{q^2-2q}^{(1)} = \frac{q^3-q}{2}, \quad A_{q^2-2q+1}^{(1)} = \frac{q^4+q^3}{2},$

$A_{q^2-q-1}^{(1)} = \frac{q^5-2q^4+q^3}{2}, \quad A_{q^2-q-1}^{(2)} = q^4 - q^2,$

$A_{q^2-q}^{(1)} = q^4 + q^2 + 2q, \quad A_{q^2-q}^{(2)} = 2q^3 + 3q^2 + q, \quad A_{q^2-q}^{(3)} = q^2 + q,$

$A_{q^2-q+1}^{(1)} = \frac{q^5-q^3}{2},$

$A_{q^2-4}^{(2)} = \frac{q^8-4q^7+5q^6+q^5-6q^4+3q^3}{24},$

$A_{q^2-3}^{(2)} = 4q^7 - 9q^6 + q^5 + 9q^4 - 5q^3, \quad A_{q^2-3}^{(3)} = \frac{q^6-q^5-q^4+q^3}{6},$

$A_{q^2-2}^{(2)} = \frac{q^8-2q^7+13q^6-9q^5-14q^4+11q^3}{4}, \quad A_{q^2-2}^{(3)} = \frac{q^7+q^5-2q^3}{2}, \quad A_{q^2-2}^{(4)} = \frac{q^4-q^2}{4},$

$A_{q^2-1}^{(1)} = \frac{q^4-q^3}{2}, \quad A_{q^2-1}^{(2)} = \frac{2q^8+4q^7-5q^6+29q^5+15q^4-27q^3+6q^2}{6},$

$A_{q^2-1}^{(3)} = \frac{2q^8+3q^6+3q^5+5q^4+3q^3}{2}, \quad A_{q^2-1}^{(4)} = q^6 + q^5 + q^3 + 2q^2, \quad A_{q^2-1}^{(5)} = q^2,$

$A_{q^2}^{(1)} = \frac{q^3-q+2}{2}, \quad A_{q^2}^{(2)} = 9q^8 + 8q^7 + 21q^6 - 19q^5 + 42q^4 + 59q^3 - 24q^2 + 24,$

$A_{q^2}^{(3)} = \frac{6q^9+9q^7+8q^6+7q^5+4q^4+14q^3+6q^2+6}{6}, \quad A_{q^2}^{(4)} = \frac{2q^8+2q^7+2q^6+2q^5+5*q^4+2q^3+q^2+2q+2}{2},$

$A_{q^2}^{(5)} = q^5 + q^4 + q^3 + q + 1, \quad A_{q^2}^{(6)} = 1$, and all other $A_w^{(r)}$ are zero.

Higher weight spectra of $\mathrm{RM}_q(2,2)$ when $q = 5,4,3,2$

- For $\mathrm{RM}_5(2,2)$, the same formulas (as in the case $q \geq 7$) work for all $w \neq q^2 - q + 1 = q^2 - 4 = 21$, and $A_{21}^{(1)} = 1500$, $A_{21}^{(2)} = 6500$ and $A_{21}^{(r)} = 0$ for $r = 3,4,5,6$.
- For $\mathrm{RM}_4(2,2)$, the same formulas work for all $w \neq q^2 - q = q^2 - 4 = 12$ and $w \neq q^2 - q + 1 = q^2 - 3 = 13$. Moreover, $A_{12}^{(1)} = 280$, $A_{12}^{(2)} = 1020$, $A_{12}^{(3)} = 20$, and $A_{12}^{(r)} = 0$ for $r = 4,5,6$. Furthermore, $A_{13}^{(1)} = 480$, $A_{13}^{(2)} = 5280$, $A_{13}^{(3)} = 480$, and $A_{13}^{(r)} = 0$ for $r = 4,5,6$.
- For $\mathrm{RM}_3(2,2)$, the same formulas work for all $w \neq q^2 - q - 1 = q^2 - 4 = 5$, $w \neq q^2 - q = q^2 - 3 = 6$, and $w \neq q^2 - q + 1 = q^2 - 2 = 7$. $A_5^{(1)} = 54$, and $A_5^{(2)} = 126$, and $A_5^{(r)} = 0$, for $r = 3,4,5,6$. Furthermore $A_6^{(1)} = 96$, and $A_6^{(2)} = 588$, and $A_6^{(3)} = 84$, and $A_6^{(r)} = 0$, for $r = 4,5,6$, and $A_7^{(1)} = 108$, and $A_7^{(2)} = 2160$, and $A_7^{(3)} = 1188$, and $A_7^{(r)} = 0$, for $r = 4,5,6$.
- For $\mathrm{RM}_2(2,2)$, $A_1^{(1)} = 4,\ A_2^{(1)} = 6,\ A_3^{(1)} = 4,\ A_4^{(1)} = 1,\ A_2^{(2)} = 6,\ A_3^{(2)} = 16,$ $A_4^{(2)} = 13,\ A_3^{(3)} = 4,\ A_4^{(3)} = 11,\ A_4^{(4)} = 1$, and all other $A_w^{(r)}$ are zero.

Idea of Proof

Some of the terminology below will be explained later. We outline here the main steps. Let $C = \mathrm{RM}_q(2,2)$ and let $n = q^2$. Note that $\dim C = 6$.

- First, solve the more general problem of determining the (graded) Betti numbers $\beta_{i,j}$ of the matroid $\mathcal{M}_C$ associated to C and also the Betti numbers $\beta_{i,j}^{(\ell)}$ of the elongations $\mathcal{M}_C^{(\ell)}$ of $\mathcal{M}_C$. for $\ell = 0, 1, \ldots, 6$.

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- Use a result of J–Roksvold–Verdure (2016), which shows that

$$P_w(T) = \sum_{\ell \geq 0} \sum_{i \geq 0} (-1)^{i+1} (\beta_{i,w}^{(\ell-1)} - \beta_{i,w}^{(\ell)}) T^l \quad \text{for } 0 \leq w \leq n,$$

  where $P_w(T)$ is the so-called generalized weight polynomial of C which is a univariate polynomial with integer coefficients having the property that $P_w(q^e)$ is the number of codewords of weight w of $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^e}$ for each $e \geq 0$.

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- Use results of Helleseth–Kløve–Mykkeltveit (1977) and Jurrius (2012) that relate the higher weight spectra with generalized weight polynomials:

$$P_w(q^e) = \sum_{r=0}^{e} A_w^{(r)} \prod_{i=0}^{r-1} (q^e - q^i) \quad \text{for } e \geq 0 \text{ and } 0 \leq w \leq n.$$

Review of Matroids and Betti numbers

Definition

Let E be a finite set. A finite matroid (or simply, matroid) on E is a pair $(E, \mathcal{I})$ where $\mathcal{I}$ is a family of subsets of E with the following properties:

(i) $\emptyset \in \mathcal{I}$,

(ii) if $\sigma \subseteq \tau$ and $\tau \in \mathcal{I}$, then $\sigma \in \mathcal{I}$,

(iii) if $\sigma, \tau \in \mathcal{I}$ with $|\sigma| < |\tau|$, then there exists $x \in \tau \setminus \sigma$ such that $\sigma \cup \{x\} \in \mathcal{I}$.

Review of Matroids and Betti numbers

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Let E be a finite set. A finite matroid (or simply, matroid) on E is a pair $(E, \mathcal{I})$ where $\mathcal{I}$ is a family of subsets of E with the following properties:
(i) $\emptyset \in \mathcal{I}$,
(ii) if $\sigma \subseteq \tau$ and $\tau \in \mathcal{I}$, then $\sigma \in \mathcal{I}$,
(iii) if $\sigma, \tau \in \mathcal{I}$ with $|\sigma| < |\tau|$, then there exists $x \in \tau \setminus \sigma$ such that $\sigma \cup \{x\} \in \mathcal{I}$.

Fix a matroid $\mathcal{M} = (E, \mathcal{I})$. Elements of $\mathcal{I}$ are called independent sets of $\mathcal{M}$.

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- Let $\sigma \subseteq E$. The rank and nullity of σ are
$$\rho(\sigma) := \max\{|\tau| : \tau \subseteq \sigma \text{ and } \tau \in \mathcal{I}\} \quad \text{and} \quad n(\sigma) := |\sigma| - \rho(\sigma).$$
- rank$(\mathcal{M})$:= $\max\{|\sigma| : \sigma \in \mathcal{I}\} = \rho(E)$.
- The i-th generalized null space of $\mathcal{M}$ is
$$N_i := \{\sigma \subseteq E : n(\sigma) = i\} \quad \text{for } i = 0, \ldots, n(E)..$$
- A cycle of $\mathcal{M}$ is an inclusion-minimal subset in N_i for some i.

Euler characteristic, Möbius function and Elongations

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid.

Definition

The Euler characteristic of $\mathcal{M}$ is $\chi(\mathcal{M}) := \sum_{i \geq 0} (-1)^{i+1} |\{\tau \in \mathcal{I} : |\tau| = i\}|$.
The Möbius function $\mu = \mu_{\mathcal{M}}$ of $\mathcal{M}$ is the $\mathbb{Z}$-valued function on the lattice $L_{\mathcal{M}}$ of cycles of $\mathcal{M}$ defined recursively by $\mu(\emptyset) = 1$ and for $\sigma \in L_{\mathcal{M}}$ with $\sigma \neq \emptyset$,

$$\mu(\sigma) = -\sum_{\tau} \mu(\tau), \text{ where the sum is over } \tau \in L_{\mathcal{M}} \text{ with } \tau \subsetneq \sigma.$$

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Definition

Let $\ell \geq 0$. Then, the ℓ-th elongation of $\mathcal{M}$ is the matroid $\mathcal{M}^{(\ell)} = (E, \mathcal{I}^{(\ell)})$ with

$$\mathcal{I}^{(\ell)} = \{I \cup \sigma : I \in \mathcal{I}, \ \sigma \subseteq E, \text{ and } |\sigma| \leq \ell\}.$$

- $\mathcal{I}^{(\ell)}$ is the set of all subsets of E if $\ell \geq n(E) = |E| - \text{rank}(\mathcal{M})$.
- $\text{rank}(\mathcal{M}^{(\ell)}) = \min\{|E|, \ \text{rank}(\mathcal{M}) + \ell\}$.
- The i-th generalized null space $N_i^{(\ell)}$ of $\mathcal{M}^{(\ell)}$ is $N_{i+\ell}$ for $i = 0, \ldots, n(E) - \ell$.

Simplicial complexes and Stanley-Reisner rings

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Let E be a finite set. A collection Δ of subsets of E is a simplicial complex if

$$\sigma \in \Delta \text{ and } \tau \subseteq \sigma \implies \tau \in \Delta.$$

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- Fix a field $\Bbbk$ and let S be the polynomial ring $\Bbbk[X_e : e \in E]$.
- The Stanley-Reisner ideal I_Δ of Δ is the ideal of S generated by the monomials corresponding to non-faces, i.e.,

$$I_\Delta = \langle \mathbf{x}^\sigma : \sigma \subseteq E \text{ and } \sigma \notin \Delta \rangle, \quad \text{where } \mathbf{x}^\sigma = \prod_{e \in \sigma} X_e \text{ for any } \sigma \subseteq E.$$

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- The Stanley-Reisner ring of Δ is

$$R_\Delta = S/I_\Delta.$$

This is clearly a S-module and a vector space over $\Bbbk$.

Betti numbers of simplicial complexes

Let $n := |E|$. Since I_Δ is a monomial ideal of S, R_Δ is a $\mathbb{N}^n$-graded finitely generated S-module. As such it has a minimal free resolution of the form

$$0 \longleftarrow R_\Delta \xleftarrow{\partial_0} S_0 \xleftarrow{\partial_1} S_1 \longleftarrow \cdots \xleftarrow{\partial_l} S_l \longleftarrow 0$$

where $S_0 = S$ and each S_i is a $\mathbb{N}^n$-graded free S-module of the form

$$S_i = \bigoplus_{\alpha \in \mathbb{N}^n} S(-\alpha)^{\beta_{i,\alpha}} = \bigoplus_{\sigma \subseteq E} S(-\sigma)^{\beta_{i,\sigma}}.$$

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Here $\beta_{i,\alpha}$ are independent of the choice of the minimal free resolution and they are called the $\mathbb{N}^n$-graded Betti numbers of Δ. For $d, i \geq 0$, we let

$$\beta_{i,d} = \sum_{|\alpha|=d} \beta_{i,\alpha} \quad \text{and} \quad \beta_i = \sum_{d \geq 0} \beta_{i,d} \quad \text{and} \quad \phi_j = \sum_i (-1)^i \beta_{i,j}.$$

We call $\beta_{i,d}$ and β_i the $\mathbb{N}$-graded and ungraded Betti numbers of Δ, respectively.

Matroids associated to linear codes

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- Let C be an $[n,k]_q$-code and let H be a parity check matrix of C.
- Let $E = [n] := \{1,2,\ldots,n\}$ and for $i \in E$, let H_i be the i-th column of H.
- Define $\Delta_C := \{\sigma \subseteq E : \{H_i : i \in \sigma\}$ is linearly independent over $\mathbb{F}_q\}$.

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- Δ_C is a simplicial complex. Let R_C be its Stanley-Reisner ring.
- R_C is Cohen-Macaulay and $\dim(R_C) = n - k$.
- By the Auslander-Buchsbaum formula, the length of any minimal free resolution of R_C is $\operatorname{depth}(S) - \operatorname{depth}(R_C)$, i.e. , $n - (n - k) = k$.
- $\beta_{i,\sigma}$, $\beta_{i,d}$ and β_i are independent of the choice of field $\mathbb{k}$ and are called the $\mathbb{N}^n$-graded, $\mathbb{N}$-graded, and ungraded Betti numbers of C, respectively.

Useful Results for solving the more general problem

- [J–Verdure (2013)] Betti numbers of C determine its generalized Hamming weights. In fact,

$$d_i(C) = \min\{j : \beta_{i,j} \neq 0\} \quad \text{for } i = 1, \ldots, k.$$

Let $\ell \geq 0$ and consider the $\mathbb{N}$-graded and $\mathbb{N}^n$-graded Betti numbers of $\mathcal{M}_C^{(\ell)}$.

- [Peskine–Szpiro (1974); Boij–Søderberg (2008); Herzog–Kühl]

$$\sum_{i\geq 0}\sum_{w\geq 0}(-1)^i w^s \beta_{i,w}^{(\ell)} = 0 \quad \text{for } 0 \leq s \leq k-1 \quad [\text{where by convention, } 0^0 = 1].$$

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$$\beta_{i,\sigma}^{(\ell)} \neq 0 \iff \sigma \text{ is inclusion-minimal in } N_i^{(\ell)}.$$

- [Hochster (1977) + Björner (1992)] If $\sigma \in N_i$, then

$$\beta_{i,\sigma}^{(\ell)} = (-1)^{r(\sigma)-1}\chi(\mathcal{M}_\sigma^{(\ell)}), \quad \text{where } \mathcal{I}(\mathcal{M}_\sigma^{(\ell)}) := \{\tau \in \mathcal{I}(\mathcal{M}^{(\ell)}) : \tau \subseteq \sigma\}.$$

- [Stanley (1977)] $\beta_{i,\sigma} = |\mu(\sigma)|$ for any inclusion-minimal element σ of N_i.

Now suppose $C = RM_q(2,2)$. Then the ground set E of the associated matroid $\mathcal{M}_C$ can be identified with $\mathbb{A}^2(\mathbb{F}_q)$. A crucial observation is the following.

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Lemma

Assume that $q \geq 3$. Then the nullity of any $\sigma \subseteq E$ is equal to the $\mathbb{F}_q$-vector space dimension of the space of affine conics in $\mathbb{A}^2(\mathbb{F}_q)$ passing through $E \setminus \sigma$:

$$n(\sigma) = \dim_{\mathbb{F}_q} \{f \in \mathbb{F}_q[X,Y]_{\leq 2} : f(P) = 0 \text{ for all } P \in E \setminus \sigma\}.$$

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Theorem (Hirschfeld)

In $\mathbb{P}^2(\mathbb{F}_q)$, the $\frac{q^6-1}{q-1}$ conics (corresponding to polynomials in $\mathbb{F}_q[X,Y,Z]_2$) are:

- q^2+q+1 *double lines with* $q+1$ *points in* $\mathbb{P}^2(\mathbb{F}_q)$,
- $\frac{q(q+1)(q^2+q+1)}{2}$ *pairs of two distinct lines with* $2q+1$ *points in* $\mathbb{P}^2(\mathbb{F}_q)$.
- q^5-q^2 *irreducible conics with* $q+1$ *points in* $\mathbb{P}^2(\mathbb{F}_q)$,
- $\frac{q(q-1)(q^2+q+1)}{2}$ *conics (pairs of Galois-conjugate lines defined over* $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$*) that just possess a single* $\mathbb{F}_q$*-rational point each.*

Classification of conics in $\mathbb{A}^2(\mathbb{F}_q)$

We use the result of Hirschfeld to work out a classification of affine conics. Denote by L, say $Z = 0$, the line at infinity in $\mathbb{P}^2(\mathbb{F}_q)$.

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Theorem

- *The* $q^2 + q + 1$ *double lines in* $\mathbb{P}^2(\mathbb{F}_q)$ *are divided into* 2 *categories:*
  a) 1 *double line* $Z^2 = 0$, *with no points in* $\mathbb{A}^2(\mathbb{F}_q)$.
  b) $q^2 + q$ *other double lines, each with* q *zeros in* $\mathbb{A}^2(\mathbb{F}_q)$.
- *The* $\frac{q(q+1)(q^2+q+1)}{2}$ *pairs of two distinct lines are divided into* 3 *categories:*
  c) $q^2 + q$ *line pairs of the type* $Z\,F(X,Y,Z) = 0$, *where* $F(X,Y,Z)$ *is a linear form not proportional to* Z. *These conics have* q *zeros in* $\mathbb{A}^2(\mathbb{F}_q)$.
  d) $\frac{q^4+q^3}{2}$ *line pairs intersecting outside* L. *These conics have* $2q-1$ *zeros in* $\mathbb{A}^2(\mathbb{F}_q)$.
  e) $\frac{q(q^2-1)}{2}$ *line pairs intersecting at a single point of the line* L. *Such line pairs have* $2q$ *zeros in* $\mathbb{A}^2(\mathbb{F}_q)$.

Classification of conics in $\mathbb{A}^2(\mathbb{F}_q)$ contd.

- The $q^5 - q^2$ irreducible conics are divided into 3 categories:
  - f) $\frac{q^3(q^2-1)}{2}$ conics intersecting L in two distinct points over $\mathbb{F}_q$. These conics have $q-1$ zeros in $\mathbb{A}^2(\mathbb{F}_q)$.
  - g) $q^2(q^2-1)$ conics being tangent to L at one $\mathbb{F}_q$-rational point. These conics have q zeros in $\mathbb{A}^2(\mathbb{F}_q)$.
  - h) $\frac{q^3(q-1)^2}{2}$ conics that have no F_q-rational point on L. These conics have $q+1$ zeros in $\mathbb{A}^2(\mathbb{F}_q)$.
- The $\frac{q(q-1)(q^2+q+1)}{2}$ conics that just possess a single $\mathbb{F}_q$-rational point each, are divided into 2 categories:
  - i) $\frac{q^2(q^2-q)}{2}$ conics, where the single point is not on L. These conics have 1 point in $\mathbb{A}^2(\mathbb{F}_q)$.
  - j) $\frac{q^3-q}{2}$ conics, where the single point is on L. These conics have 0 points in $\mathbb{A}^2(\mathbb{F}_q)$.

Using the Classification to complete the quest

- In the above classification, cases d), e), f), g), h) correspond to the minimal codewords of $C = \mathrm{RM}_q(2,2)$. These give the values of all the $\beta_{1,j}$.

- We further determine the minimal sets in $N_1^{(\ell)} = N_{1+\ell}$ to get the values of all the $\beta_{1,j}^{(\ell)}$.

- Additionally, we determine some more $\beta_{i,j}^{(\ell)}$ for $i \geq 2$ partly by using the Boij-Søderberg equations.

- We then go back to the relation between the Betti numbers of (the elongations of matroids corresponding to) C to determine the generalized weight polynomials.

- In turn, the generalized weight polynomials are used to obtain the higher weight spectra.

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Problem for Brave People: Do this for any $\mathrm{RM}_q(r,m)$ and $\mathrm{PRM}_q(r,m)$.

Thanks for your attention!

Reference: S. R. Ghorpade, T. Johnsen, R. Ludhani, and R. Pratihar, Higher weight spectra and Betti numbers of Reed-Muller codes $\mathrm{RM}_q(2,2)$, https://arxiv.org/html/2408.02548v1

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