The incidence matrix of a q-ary graph

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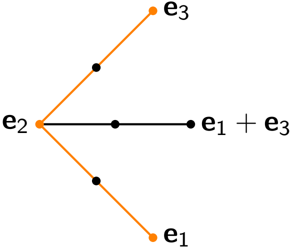
Definition

Let $V = \mathbb{F}_q^v$ and let E be a set of 2-dimensional subspaces of V, the *edges*. Then (V, E) is a q-**ary graph** if for all $c_1, c_2 \in \mathbb{F}_q$:

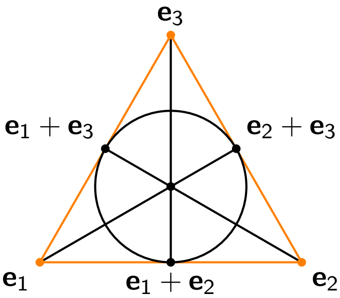
If $\langle \mathbf{x}, \mathbf{y_1} \rangle$ *and* $\langle \mathbf{x}, \mathbf{y_2} \rangle$ *are (adjecent) edges, then* $\langle \mathbf{x}, c_1\mathbf{y_1} + c_2\mathbf{y_2} \rangle$ *is an edge.*

In other words: neighbourhoods are spaces.

Example



q-ary P_2 in $\mathbb{F}_2^3$



q-ary C_3 in $\mathbb{F}_2^3$

Question

Do q-ary graphs have a nice geometric interpretation?

Question

Can we "q-ify" each graph?

Incidence matrix of a graph: matrix over $\mathbb{F}_q$ (usually $\mathbb{F}_2$) with v rows such that

- columns $\leftrightarrow$ edges;
- Hamming support $=$ edge (as a set of vertices);
- so: Hamming weight $= 2$;
- orthogonal to $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\mathsf{T}$ (full Hamming weight vector).

Incidence matrix of a q-ary graph: matrix over $\mathbb{F}_{q^v} = \mathbb{F}_q[\alpha]$ with v rows such that

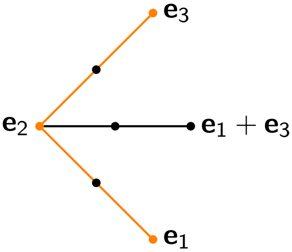
- columns $\leftrightarrow$ edges;
- rank support $=$ edge (as a space);
- so: rank weight $= 2$;
- orthogonal to $\begin{bmatrix} 1 & \alpha & \cdots & \alpha^{v-1} \end{bmatrix}^\mathsf{T}$ (full rank weight vector);
- behaves nicely with the q-ary graph property.

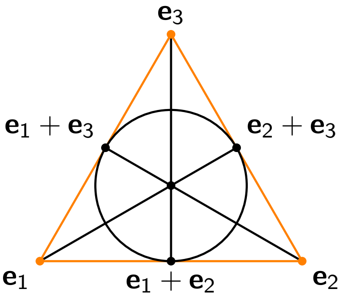
Theorem

Let $\langle \mathbf{x}, \mathbf{y}_1 \rangle, \ldots, \langle \mathbf{x}, \mathbf{y}_d \rangle$ be edges through the same vertex. Fix a representation $\mathbf{v}_1$ of the edge $\langle \mathbf{x}, \mathbf{y}_1 \rangle$. Then there exist unique representations $\mathbf{v}_2, \ldots, \mathbf{v}_d$ of the edges $\langle \mathbf{x}, \mathbf{y}_2 \rangle, \ldots, \langle \mathbf{x}, \mathbf{y}_d \rangle$ such that for any $\lambda_1, \ldots, \lambda_d \in \mathbb{F}_q$ the vector $\mathbf{v}_{d+1} := \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_d \mathbf{v}_d$ is a representation of the edge $\langle \mathbf{x}, \sum_{i=1}^{d} \lambda_i \mathbf{y}_i \rangle$.

Proof: constructive, by linear algebra.

Example





$$\begin{bmatrix} \alpha & 0 & \alpha \\ 1 & \alpha^2 & \alpha^6 \\ 0 & \alpha & \alpha \end{bmatrix}$$

$$\begin{bmatrix} \alpha & 0 & \alpha^2 & \alpha & \alpha^4 & \alpha^2 & \alpha^4 \\ 1 & \alpha^2 & 0 & \alpha^6 & 1 & \alpha^2 & \alpha^6 \\ 0 & \alpha & 1 & \alpha & 1 & \alpha^3 & \alpha^3 \end{bmatrix}$$

Theorem

For every q-ary graph, fixing the representation of one edge fixes a representation for all other edges up to a scalar in $\mathbb{F}_q^*$.
Starting with a different representation for the first edge, or with a different first edge, will multiply the whole incidence matrix with a scalar $\mathbb{F}_{q^v}^*$.

So: different incidence matrices give isomorphic q-matroids.

Concluding remarks:

- We can make a q-matroid from a q-ary graph!
- We can motivate this incidence matrix by doing geometry over $\mathbb{F}_1$.
- How to get directly from a q-ary graph to a q-matroid? Still no idea!

Wild speculation:

- We made a q-analogue of a characteristic vector. Can this be extended to other applications, like polytopes?
- Maybe we can make the definition of a q-ary SRG less strict?

Thank you for your attention!

arxiv.org/abs/2508.19964