Designs of Perfect Matchings

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Paderborn University

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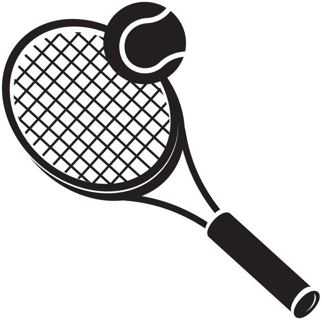
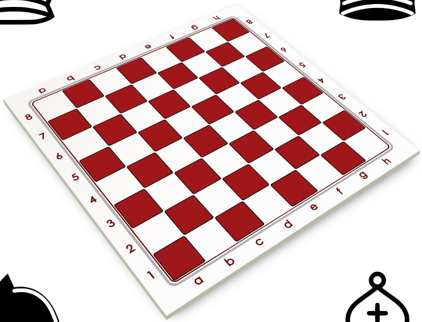
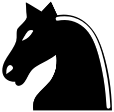
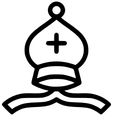
Games!









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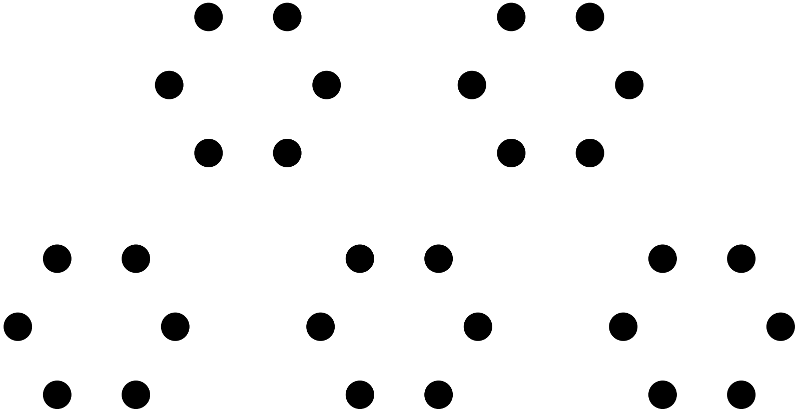

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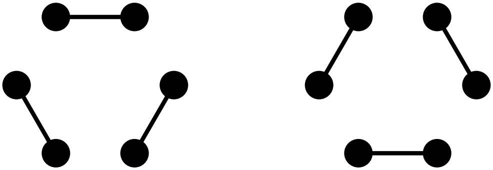

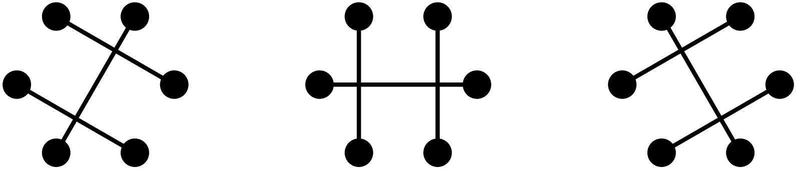

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Generalisation:

- perfect matching $\longrightarrow$ uniform set partition
- pair of disjoint subsets $\longrightarrow$ t disjoint subsets

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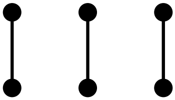

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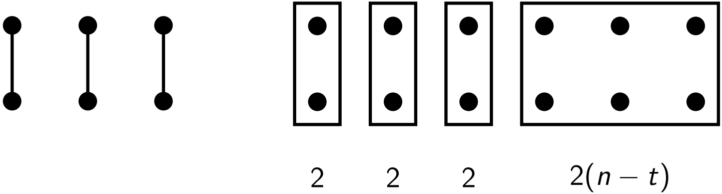
t disjoint edges $\longrightarrow$ set partition of shape $(2(n-t), 2, 2, \ldots, 2)$



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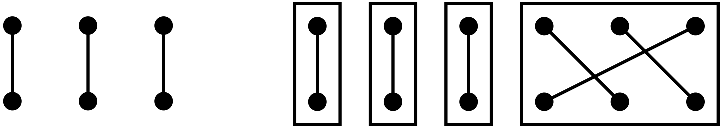

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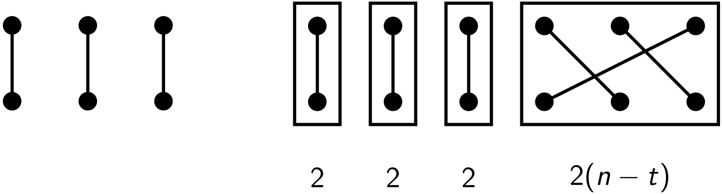

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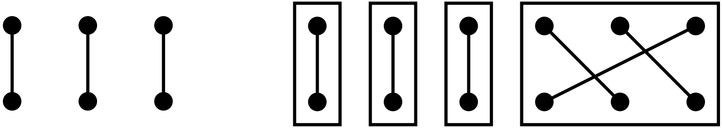

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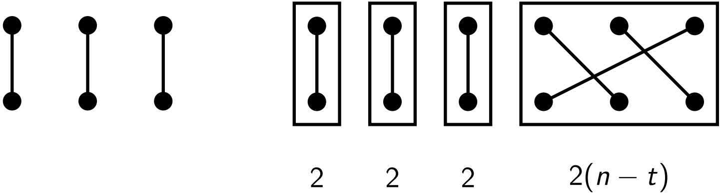

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$\rightsquigarrow$ 1-factorisation: $(n-1, 1)$-factorisation
hyperfactorisation: $(n-2, 1, 1)$-factorisation

Example

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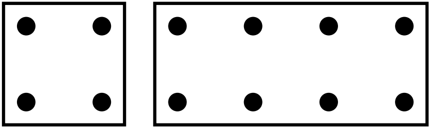

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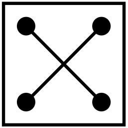

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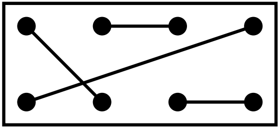

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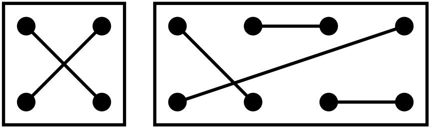

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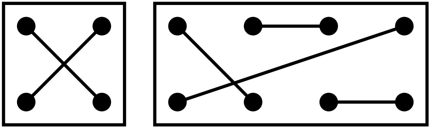

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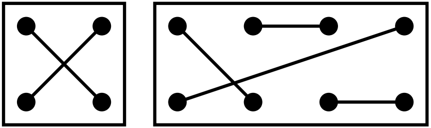

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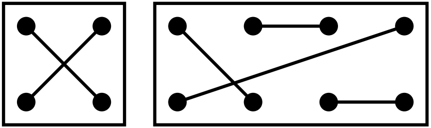

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$D = M_1^G \cup M_2^G$ is (42)-factorisation of index 1

Main results

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Theorem [Bamberg, K., (Schmidt) 2025]

Let $D \subseteq \mathcal{M}_{2n}$ be a non-empty set of perfect matchings and $(a'_\mu)_{\mu \vdash n}$ be its dual distribution. Then

$$D \text{ is a } \lambda\text{-factorisation} \iff a'_\mu = 0 \text{ for all } \mu \vdash n \text{ with } \lambda \trianglelefteq \mu \neq (n).$$

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Theorem [Bamberg, K., (Schmidt) 2025]

Let $D \subseteq \mathcal{M}_{2n}$ be a λ-factorisation. If $\mu \trianglerighteq \lambda$, then D is also μ-factorisation.

Construction

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Cameron: Hyperovals in finite projective planes

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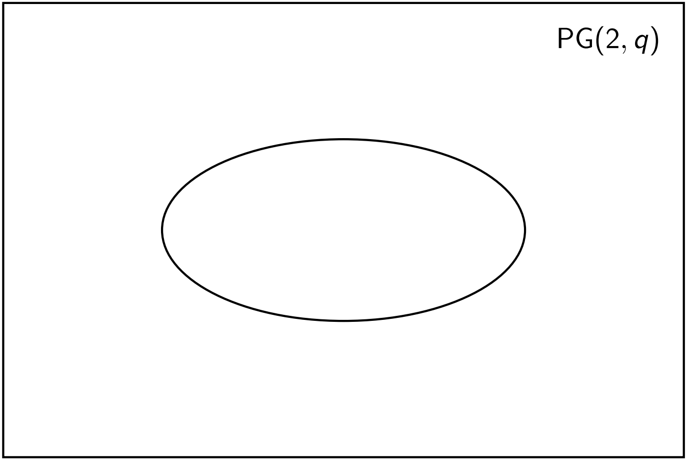

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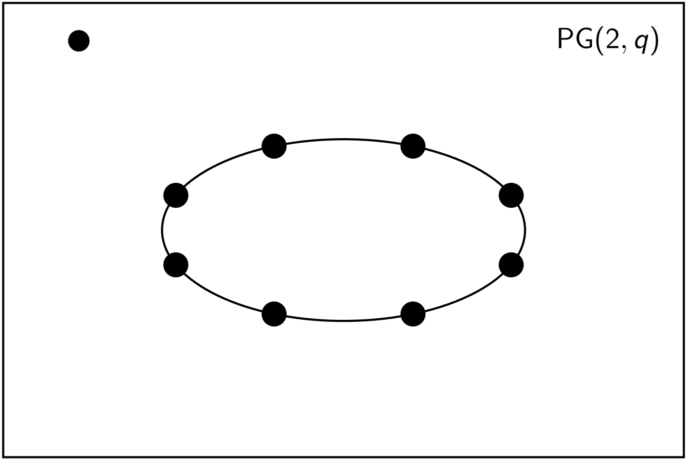

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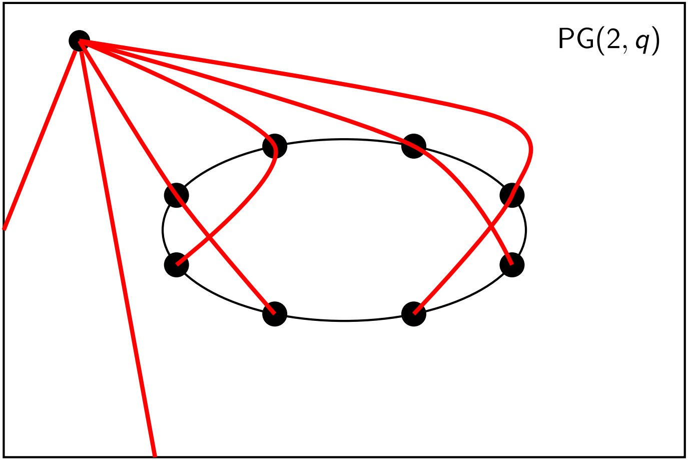

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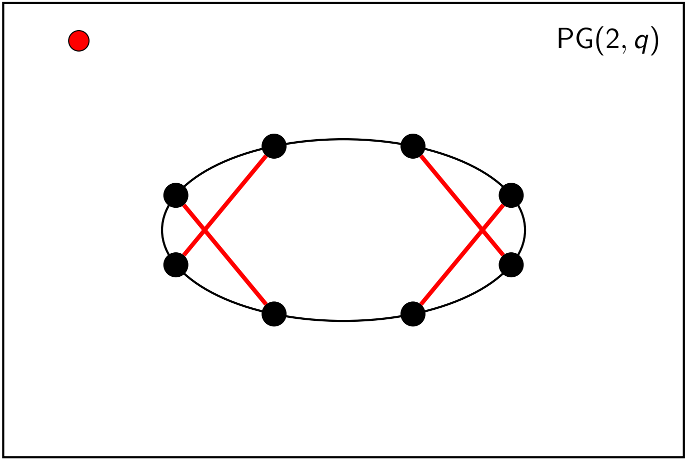

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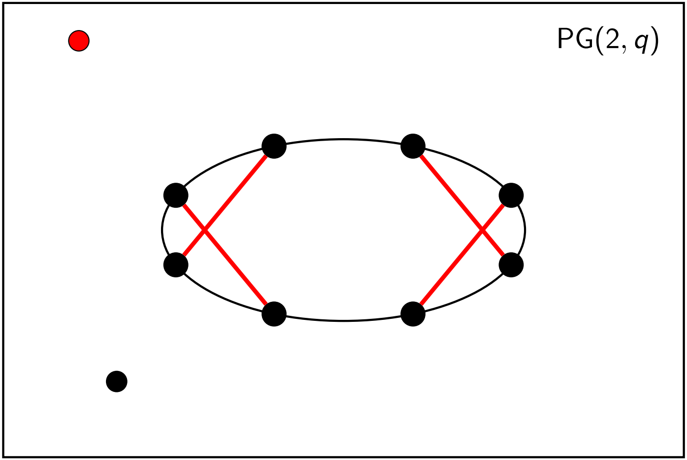

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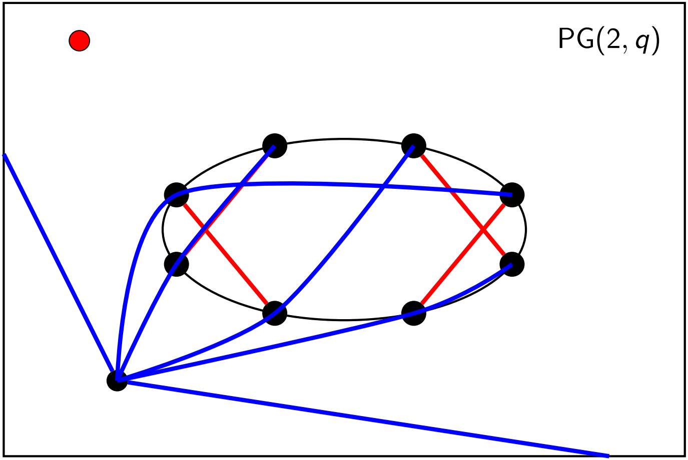

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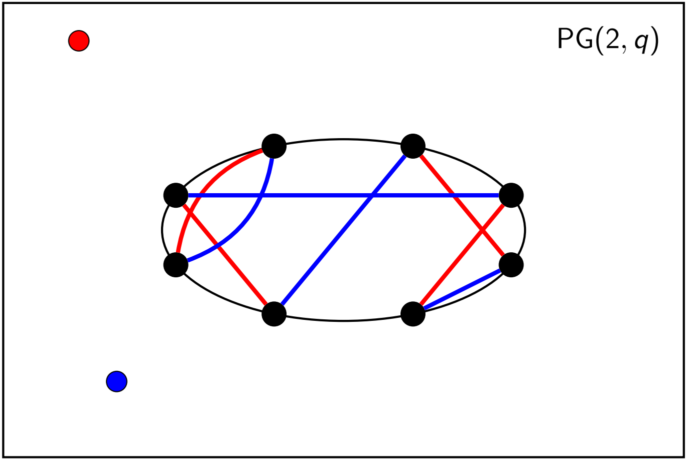

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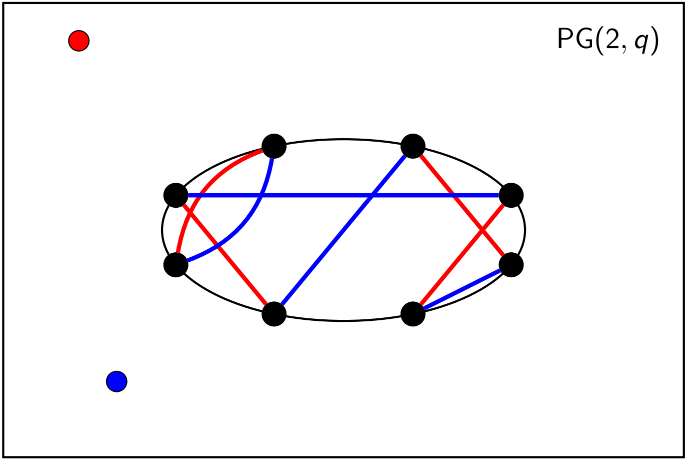

$\longrightarrow$ hyperfactorisation on points of the oval

Thank you for your attention!