Additive codes attaining the Griesmer bound

Sascha Kurz (University of Bayreuth)

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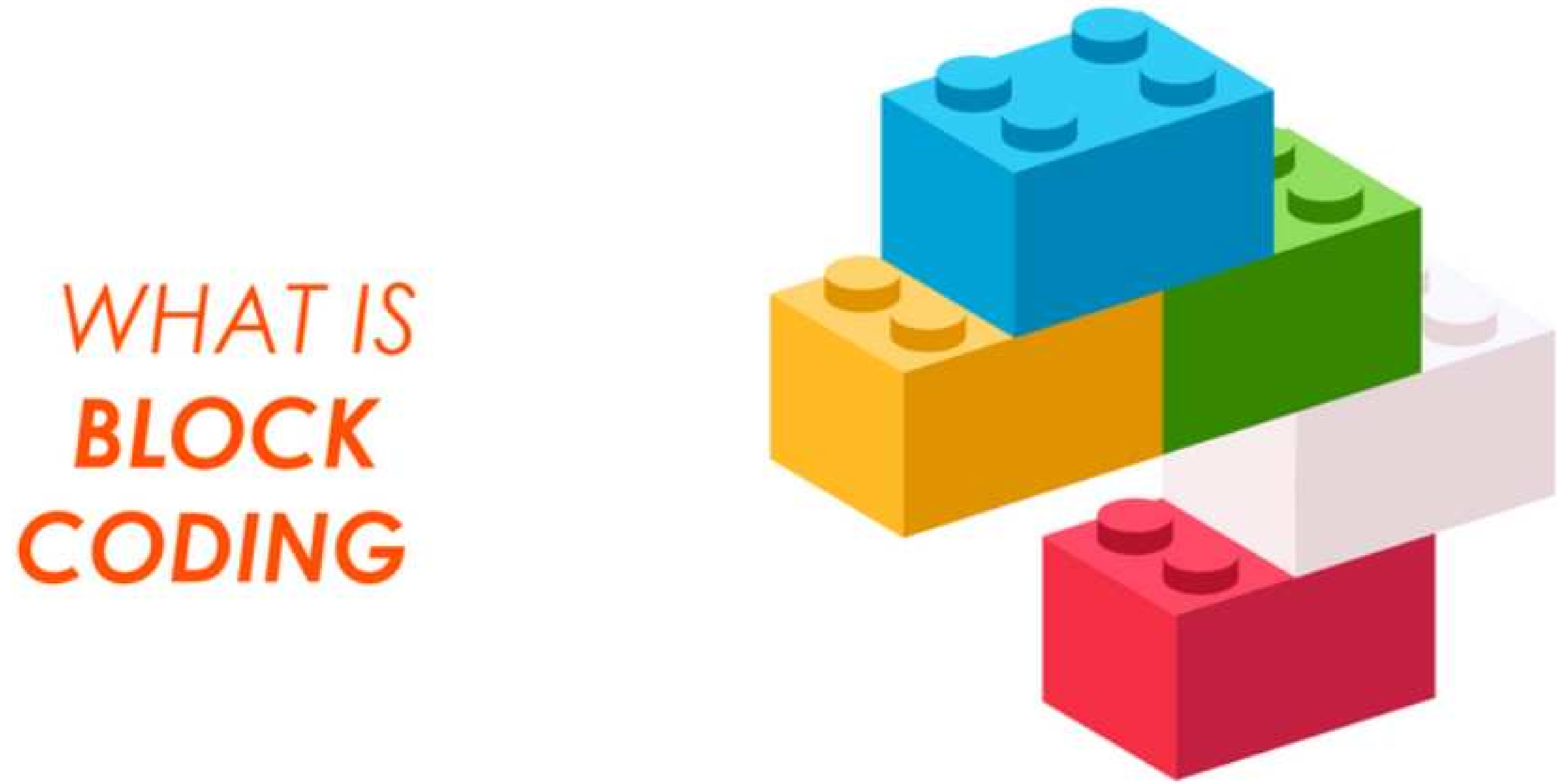
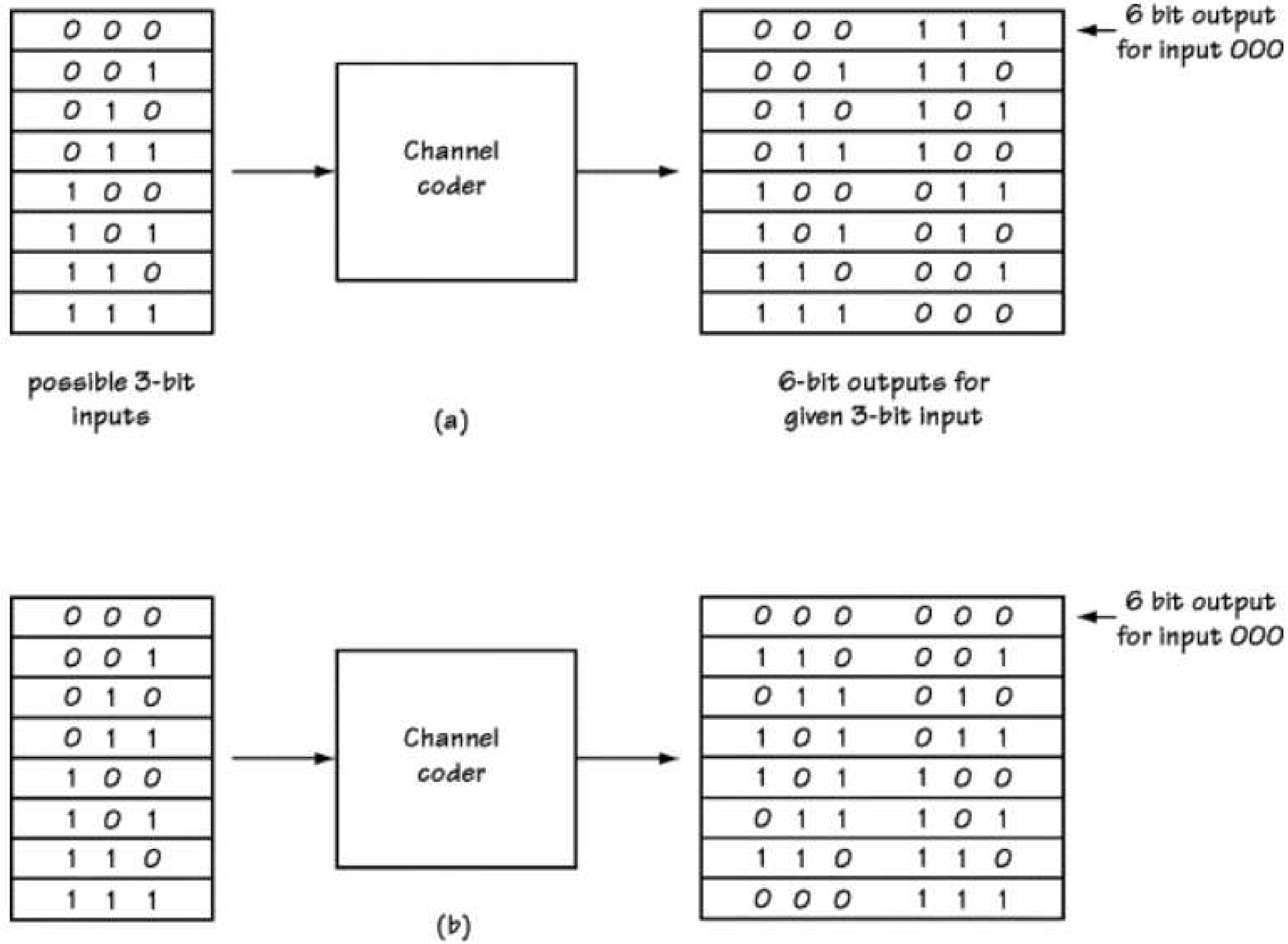
Introduction



Block coding



Block codes

- alphabet $\mathcal{A}$
- length n
- block code $C \subseteq \mathcal{A}^n$
- metric d on $\mathcal{A}^n$
- minimum distance $d(C) := \min\{d(c, c') \mid c, c' \in C, c \neq c'\}$

Block codes

- alphabet $\mathcal{A}$
  $\mathcal{A} = \{a, e, g, i, m, l, r, C, E, T\}$
- length n
  $n = 5$
- block code $C \subseteq \mathcal{A}^n$
  $C = \{Camel, Eagle, Tiger\}$
- metric d on $\mathcal{A}^n$
  Hamming distance: $d(Camel, Eagle) = 4$, $d(Camel, Tiger) = 4$, $d(Eagle, Tiger) = 4$
- minimum distance $d(C) := \min\{d(c, c') \mid c, c' \in C, c \neq c'\}$
  $d(C) = \min\{4, 4, 4\} = 4$

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Let $A_q(n, d)$ be the maximum size of a block code with codewords of length n and minimum distance d over an alphabet of size q. $\rightsquigarrow$ determination of $A_q(n, d)$

$A_2(10,3) \geq 72$

$$
\begin{aligned}
C \;=\; \{&0000000000, 1110100000, 0011100000, 1010010000, 0101110000, 0110001000,\\
&1001101000, 0001011000, 1111011000, 1100111000, 1011000100, 0100100100,\\
&1100010100, 0111010100, 1001110100, 1000001100, 0101001100, 1111101100,\\
&0010011100, 1101100010, 0110010010, 0000110010, 1011110010, 0011001010,\\
&0100101010, 1010101010, 1000011010, 0111111010, 0010000110, 1000100110,\\
&0111100110, 0001010110, 1110110110, 1110001110, 0001101110, 0100011110,\\
&1011011110, 1101111110, 0111000001, 1000100001, 0100010001, 1001010001,\\
&1111110001, 1100001001, 1011001001, 0101101001, 0000111001, 1110000101,\\
&0001000101, 0010100101, 1101100101, 1101011101, 0110111101, 1010000011,\\
&0110100011, 0001100011, 0011010011, 1100110011, 0000001011, 1111101011,\\
&1110011011, 0101011011, 1001111011, 0100000111, 1011100111, 1000010111,\\
&1111010111, 0101110111, 1001001111, 0111001111, 1100101111, 0011111111\}
\end{aligned}
$$

$A_2(10,3) = 72$

$$\begin{aligned} C = \{ & 0000000000, 1110100000, 0011100000, 1010010000, 0101110000, 0110001000, \\ & 1001101000, 0001011000, 1111011000, 1100111000, 1011000100, 0100100100, \\ & 1100010100, 0111010100, 1001110100, 1000001100, 0101001100, 1111101100, \\ & 0010011100, 1101100010, 0110010010, 0000110010, 1011110010, 0011001010, \\ & 0100101010, 1010101010, 1000011010, 0111111010, 0010000110, 1000100110, \\ & 0111100110, 0001010110, 1110110110, 1110001110, 0001101110, 0100011110, \\ & 1011011110, 1101111110, 0111000001, 1000100001, 0100010001, 1001010001, \\ & 1111110001, 1100001001, 1011001001, 0101101001, 0000111001, 1110000101, \\ & 0001000101, 0010100101, 1101100101, 1101011101, 0110111101, 1010000011, \\ & 0110100011, 0001100011, 0011010011, 1100110011, 0000001011, 1111101011, \\ & 1110011011, 0101011011, 1001111011, 0100000111, 1011100111, 1000010111, \\ & 1111010111, 0101110111, 1001001111, 0111001111, 1100101111, 0011111111\} \end{aligned}$$

More structure needed.

- Chapter 2, Paragraph 17 of F.J. MacWilliams and N.J.A. Sloane, The theory of error-correcting codes (1977).
- P.R.J. Östergård, T. Baicheva, and E. Kolev, Optimal binary one-error-correcting codes of length 10 have 72 codewords, IEEE Trans. Inform. Theory 45 (1999) 1229–1231. $\rightsquigarrow$ 562 non-isomorphic optimal codes

Structured codes

Let $\mathcal{A} = \mathbb{F}_q$ be a finite field and $C \subseteq \mathcal{A}^n$ be a block code.

- C is an additive code iff C is additively closed, i.e. $c, c' \in C$ implies $c + c' \in C$.
- C is a linear code iff C is linearly closed, i.e. $c, c' \in C$ and $\alpha, \alpha' \in \mathbb{F}_q$ imply $\alpha c + \alpha c' \in C$.

> Each additive code is $\mathbb{F}_{q'}$-linear over some subfield $\mathbb{F}_{q'}$, i.e. $c, c' \in C$ and $\alpha, \alpha' \in \mathbb{F}_{q'}$ imply $\alpha c + \alpha c' \in C$.

> S. Ball and T. Popatia, Additive codes from linear codes, arXiv preprint 2506.03805 (2025): “Additive codes have become of increasing importance in the field of quantum error-correction due to their equivalence to subgroups of the Pauli group and also in the field of classical error-correction, as they can provide examples of codes which outperform linear codes. It is perhaps surprising that additive codes have not been more widely studied until recently.”

Linear Codes

Definition: An $[n, k, d]_q$ code C is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance d.

Example: A $[7, 3, 4]_2$ simplex code is given by the generator matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

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Columns of a generator matrix of an $[n, k, d]_q$ code generate n points in $\mathrm{PG}(k-1, q)$. Codewords correspond to hyperplanes and the Hamming weight of the codeword equals the number of points that are not contained in the hyperplane, i.e. each hyperplane contains at most $n - d$ points.

A multiset of points $\mathcal{M}$ is a map $\mathcal{P} \to \mathbb{N}$ mapping points to multiplicities. $\mathcal{M}$ is extended additively to subspaces.

Example (cont.): A $[7, 3, 4]_2$ simplex code corresponds to the set of all seven points in $\mathrm{PG}(2, 2)$, where at most 3 are contained in a hyperplane.

The geometric point of view

Linear codes are multisets of points in $\mathrm{PG}(k-1,q)$ with at most s points in a hyperplane.

Let S_i denote an i-dimensional subspace in $\mathrm{PG}(k-1,q)$ and χ_{S_i} its characteristic function, i.e., $\chi_{S_i}(P)=1$ if $P\le S_i$ and $\chi_{S_i}(P)=0$ otherwise. Note that each hyperplane intersects an i-dimensional subspace in either dimension i or dimension $i-1$.

Example: The multiset of points $\sigma\cdot\chi_{S_k}$ in $\mathrm{PG}(k-1,q)$ corresponds to an $\left[\sigma\cdot\frac{q^k-1}{q-1},k,\sigma\cdot q^{k-1}\right]_q$ code.

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Solomon–Stiffler construction: The multiset of points $\sigma \cdot \chi_{S_k} - \sum_{i=1}^{k-1} \varepsilon_i \cdot \chi_{S_i}$ in $\operatorname{PG}(k-1,q)$ corresponds to an $\left[\sigma \cdot \frac{q^k-1}{q-1} - \sum\limits_{i=1}^{k-1} \varepsilon_i \cdot \frac{q^i-1}{q-1}, k, \sigma \cdot q^{k-1} - \sum\limits_{i=1}^{k-1} \varepsilon_i \cdot q^{i-1}\right]_q$ code provided that σ is sufficiently large.

A natural generalization

Definition (Ball, Lavrauw, Popatia): A projective $h-(n,r,s)_q$ system is a multiset $\mathcal{S}$ of n subspaces of $\mathrm{PG}(r-1,q)$ of dimension at most h such that each hyperplane contains at most s elements of $\mathcal{S}$ and some hyperplane contains exactly s elements of $\mathcal{S}$. We say that $\mathcal{S}$ is faithful if all elements have dimension h.

Remark: A multiset of points is a faithful projective $1-(n,r,s)_q$ system.

Example: A spread of h-spaces in $\mathrm{PG}(2h-1,q)$ is a faithful projective $h-\left(q^h+1,2h,1\right)_q$ system. If h divides r, then h-spreads attain the upper bound $n \le \frac{q^r-1}{q^{r-h}-1} \cdot s$ for projective $h-(n,r,s)_q$ systems.

S. Ball, M. Lavrauw, and T. Popatia, Griesmer type bounds for additive codes over finite fields, integral and fractional MDS codes, Designs, Codes and Cryptography, 93(1), 175-196 (2025).

A. Blokhuis and A.E. Brouwer, Small additive quaternary codes, European Journal of Combinatorics, 25(2), 161-167 (2004).

Additive codes

Definition: An additive $[n, r/h, d]_q^h$ code C is a subset of $\mathcal{A}^n$, where $\mathcal{A} = \mathbb{F}_{q^h}$, that is $\mathbb{F}_q$-linear, has minimum Hamming distance d, and cardinality q^r, so that $r/h \in \mathbb{Q}$ is called the dimension of C.

Observation: C can be written as the $\mathbb{F}_q$-space spanned by the rows of an $r \times n$ matrix G with entries in $\mathbb{F}_{q^h}$ $\rightsquigarrow$ generator matrix G

Construction: Let $\mathcal{B}$ be a basis for $\mathbb{F}_{q^h}$ over $\mathbb{F}_q$ and write out the elements of G over the basis $\mathcal{B}$ to obtain an $r \times nh$ matrix $\widetilde{G}$ with entries from $\mathbb{F}_q$. By $\mathcal{X}_G(C)$ we define the multiset of the n subspaces spanned by the n blocks of h columns of $\widetilde{G}$.

Theorem (Ball, Lavrauw, Popatia): If C is an additive $[n, r/h, d]_q^h$ code with generator matrix G, then $\mathcal{X}_G(C)$ is a projective $h-(n, r, n-d)_q$ system $\mathcal{S}$, and conversely, each projective $h-(n, r, s)_q$ system $\mathcal{S}$ defines an additive $[n, r/h, n-s]_q^h$ code C.

Additive codes (example)

Write $\mathbb{F}_4 \simeq \mathbb{F}_2[\omega]/\left(\omega^2+\omega+1\right)$ and consider the linear code C with generator matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \end{pmatrix}.$$

It can be easily checked that C is a $[5,2,4]_4$ code. If we interprete C as an $[5,4/2,4]_4^2$ additive code a generator matrix is e.g. given by

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & \omega & \omega & \omega & \omega \\ 1 & 0 & 1 & \omega & \omega^2 \\ \omega & 0 & \omega & \omega^2 & 1 \end{pmatrix}.$$

Here we have

$$\widetilde{G} = \begin{pmatrix} 00 & 10 & 10 & 10 & 10 \\ 00 & 01 & 01 & 01 & 01 \\ 10 & 00 & 10 & 01 & 11 \\ 01 & 00 & 01 & 11 & 10 \end{pmatrix}$$

choosing the basis $\mathcal{B} = (1,\omega)$ and using $\omega^2 = 1+\omega$.

Griesmer bound

The parameters of an $[n,k,d]_q$ code C are related by the so-called *Griesmer bound*

$$n \ge \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil =: g_q(k,d). \tag{1}$$

Interestingly enough, this bound can always be attained with equality if the minimum distance d is sufficiently large and a nice geometric construction was given by Solomon and Stiffler:

$$\sigma \cdot \chi_{S_k} - \sum_{i=1}^{k-1} \varepsilon_i \cdot \chi_{S_i} \to \left[\sigma \cdot \frac{q^k-1}{q-1} - \sum_{i=1}^{k-1} \varepsilon_i \cdot \frac{q^i-1}{q-1}, k, \sigma \cdot q^{k-1} - \sum_{i=1}^{k-1} \varepsilon_i \cdot q^{i-1} \right]_q$$

Parameterization: Write d as $d = \sigma q^{k-1} - \sum\limits_{i=1}^{k-1} \varepsilon_i \cdot q^{i-1}$, where $\sigma \in \mathbb{N}_0$ and the $0 \le \varepsilon_i < q$ are integers for all $1 \le i \le k-1$. Then, $n = g_q(k,d)$ iff $n = \sigma \cdot \frac{q^k-1}{q-1} - \sum\limits_{i=1}^{k-1} \varepsilon_i \cdot \frac{q^i-1}{q-1}$.

Griesmer bound for additive codes

Via the chain $[n, r/h, d]_q^h$ code $\to$ projective $h-(n, r, n-d)_q$ system $\to$ multiset of points $\to \left[\frac{q^h-1}{q-1} \cdot n, r, q^{h-1} \cdot d\right]_q$ code we can transfer the Griesmer bound

Lemma: To each faithful projective $h-(n, r, n-d)_q$ system we can associate a q^{h-1}-divisible $\left[n \cdot \frac{q^h-1}{q-1}, r, d \cdot q^{h-1}\right]_q$ code with maximum weight at most $n \cdot q^{h-1}$.

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Corollary: Each $[n, r/h, d]_q^h$ code satisfies

$$n \ge \left\lceil \frac{g_q\left(r, d\cdot q^{h-1}\right)\cdot(q-1)}{q^h-1} \right\rceil = \left\lceil \frac{(q-1)\cdot\sum\limits_{i=0}^{r-1}\left\lceil d\cdot q^{h-1-i}\right\rceil}{q^h-1} \right\rceil. \tag{2}$$

Interestingly enough, this bound can always be attained with equality if the minimum distance d is sufficiently large.

Partitions of multisets into h-spaces

Definition: Let $\mathcal{M}$ be a multiset of points in $\mathrm{PG}(r-1,q)$. We say that $\mathcal{M}$ is h-partitionable if there exist h-spaces $S_1,\dots,S_l$ such that $\mathcal{M}=\sum_{i=1}^{l}\chi_{S_i}$.

Observation: If $\mathcal{M}$ is h-partitionable, then $|\mathcal{M}|$ is divisible by $\frac{q^h-1}{q-1}$ and $\mathcal{M}$ is q^{h-1}-divisible, i.e. $|\mathcal{M}|\equiv|\mathcal{M}(H)|\pmod{q^{h-1}}$ for every hyperplane H.

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Definition: Let $\mathcal{M}$ be a multiset of points in $\mathrm{PG}(r-1,q)$ and $S_1\le S_2\le\dots\le S_r$ with $\dim(S_i)=i$. We say that $\mathcal{M}$ has type $\sigma[r]-\sum_{i=1}^{r-1}\varepsilon_i[i]$ iff $\mathcal{M}=\sigma\chi_{S_r}-\sum\limits_{i=1}^{r-1}\sigma_i\chi_{S_i}$, where $\sigma\in\mathbb{N}$ and $\varepsilon_i\in\mathbb{Z}$ for $1\le i\le r-1$. We say that $\sigma[r]-\sum\limits_{i=1}^{r-1}\varepsilon_i[i]$ is h-partitionable iff a multiset of points in $\mathrm{PG}(r-1,q)$ with type $\sigma[r]-\sum\limits_{i=1}^{r-1}\varepsilon_i[i]$ exists.

Observation: If $\sigma[r]-\sum\limits_{i=1}^{r-1}\varepsilon_i[i]$ is h-partionable, then the parameters of a corresponding projective $h-(n,r,s)_q$ system can be computed from σ and the ε_i.

Main result

Theorem: Let q be a prime power, $r > h \ge 1$, $g := \gcd(r,h)$, and $\varepsilon_1, \dots, \varepsilon_{r-1} \in \mathbb{Z}$ such that q^{h-i} divides ε_i for all $1 \le i < h$ and

$$\sum_{i=1}^{r-1} \varepsilon_i \cdot \frac{q^i-1}{q-1} \equiv 0 \pmod{\frac{q^g-1}{q-1}}. \tag{3}$$

Then there exists a $\sigma \in \mathbb{N}$ such that

$$\left(\sigma + t \cdot \frac{q^h-1}{q^g-1}\right)[r] - \sum_{i=1}^{r-1} \varepsilon_i [i]$$

is h-partitionable over $\mathbb{F}_q$ for all $t \in \mathbb{N}$.

Corollary: The Griesmer bound $n \ge \left\lceil \frac{g_q\left(r, d\cdot q^{h-1}\right)\cdot(q-1)}{q^h-1} \right\rceil = \left\lceil \frac{(q-1)\cdot\sum_{i=0}^{r-1}\left\lceil d\cdot q^{h-1-i}\right\rceil}{q^h-1} \right\rceil$ for $[n, r/h, d]_q^h$ codes can be attained with equality if d is sufficiently large.

Exact values

Definition: Let $n_q(r,h;s)$ denote the maximum cardinality n of a projective $h-(n,r,s)_q$ system.

Remark: $n_2(r,2;s)$ is completely determined for all $r\le 7$. For $n_2(8,2;s)$ just three values are currently unknown.

J. Bierbrauer, S. Marcugini, and F. Pambianco, Optimal additive quaternary codes of low dimension, IEEE Transactions on Information Theory, 67(8), 5116-5118 (2021).

S. K., Optimal additive quaternary codes of dimension 3.5, arXiv preprint 2410.07650, 16 pages (2024).

Definition:

$$\overline{n}_q(r,h;s) := n_{q^h}(\lceil r/h \rceil, 1; s) \tag{4}$$

In words, $\overline{n}_q(r,h;s)$ is the size of the largest projective $h-(n,r,s)_q$ system that we can naturally obtain starting from a linear code over $\mathbb{F}_{q^h}$.

Improvements

Whenever $\overline{n}_q(r,h;s) < n_q(r,h;s)$ we say that additive codes outperform linear codes for the corresponding parameters, which is especially interesting if r/h is integral.

q	r	h	s	$n_q(r,h;s)$	$\overline{n}_q(r,h;s)$
2	8	2	9	33	31
2	8	2	10	36	34
2	8	2	11	40	39
2	8	2	14	54	50
2	8	2	27	107	103
3	**6**	**2**	**3**	**21**	**17**
3	6	2	8	66–68	65

F. De Clerck, M. Delanote,N. Hamilton, and R. Mathon, Perp-systems and partial geometries, Advances in Geometry, 2(1), 1–12 (2002).

S. K., Additive codes attaining the Griesmer bound, arXiv preprint 2412.14615, 100 pages (2024).

Parametric improvements

q	r	h	i	$s_{i,t}$	$n_q(r,h;s_{i,t})$	$n_q(r,h;s_{i,t}) - \overline{n}_q(r,h;s_{i,t})$
2	8	2	13	$21t-13$	$85t-55$	2
2	8	2	14	$21t-14$	$85t-60$	2
2	8	2	18	$21t-18$	$85t-76$	2
2	8	2	19	$21t-19$	$85t-81$	2
3	6	2	7	$10t-7$	$91t-67$	3
3	6	2	8	$10t-8$	$91t-77$	3
3	6	2	9	$10t-9$	$91t-87$	3
2	9	3	5	$9t-5$	$73t-43$	2
2	9	3	6	$9t-6$	$73t-52$	2
2	9	3	7	$9t-7$	$73t-59$	4
2	9	3	8	$9t-8$	$73t-68$	4
4	6	2	9	$17t-9$	$273t-149$	4
4	6	2	10	$17t-10$	$273t-166$	4
4	6	2	11	$17t-11$	$273t-183$	4
4	6	2	12	$17t-12$	$273t-200$	4
4	6	2	13	$17t-13$	$273t-213$	8
4	6	2	14	$17t-14$	$273t-230$	8
4	6	2	15	$17t-15$	$273t-247$	8
4	6	2	16	$17t-16$	$273t-264$	8

Partitions by h-spaces

For some (pre-) multiset of points $\mathcal{M}$ we say that $\star - \mathcal{M}$ is h-partitionable in $\mathrm{PG}(r-1,q)$ iff there exists a projective $h-(n,r,s)_q$ system with type $\sigma[r] - \mathcal{M}$ for some sufficiently large $\sigma \in \mathbb{N}$.

Remark: The parameters n and s can be computed from r, σ, and $\mathcal{M}$. Assuming that σ is sufficiently large, the set of the feasible σ's is given by some explicit modulo condition. The conditions on $\mathcal{M}$ can be written down quite explicitely.

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Application: Let $A_q(r,2h;h)$ denote the maximum cardinality of a partial spread $\mathcal{P}$ of h-spaces in $\mathrm{PG}(r-1,q)$ and $\mathcal{M}$ denote the set of uncovered points. In our notation $\mathcal{P}$ is a faithful projective $h-(\#\mathcal{P},r,s,1)_q$ system $\mathcal{S}$ with type $1\cdot[r]-\mathcal{M}$, where $\#\mathcal{P}$ and s can be computed from $\mathcal{M}$. (Every point is contained in at most $\mu=1$ elements from $\mathcal{S}$.)

- $129 \le A_2(11,8;4) \le 132$: $\#\mathcal{M} \equiv 7 \pmod{15}$, $\mathcal{M}$ is 8-divisible, $\sigma \in \mathbb{N}$
  For $\# = 132$ several 8-divisible point sets of cardinality 67 exist in $\mathrm{PG}(10,2)$.
- $244 \le A_3(8,6;3) \le 248$: $\#\mathcal{M} \equiv 4 \pmod{13}$, $\mathcal{M}$ is 9-divisible, $\sigma \in \mathbb{N}$
  For $\# = 248$ there exists a unique 9-divisible point set of cardinality 56 in $\mathrm{PG}(7,3)$, the *Hill cap*.

The determination of the smallest possible σ seems to be a really hard problem.

Linear codes in the b-symbol metric

In storage applications the reading device is sometimes insufficient to isolate adjacent symbols, which makes it necessary to adjust the standard coding-theoretic error model. Cassuto and Blaum studied a model where pairs of adjacent symbols are read in every step and introduced the so-called symbol-pair metric for codes. This notion was generalized to the b-symbol metric where b-tuples of adjacent symbols are read at every step.

Y. Cassuto and M. Blaum, Codes for symbol-pair read channels, IEEE Transactions on Information Theory, 57(12), 8011-8020 (2011).

E. Yaakobi, J. Bruck, and P.H. Siegel, Constructions and decoding of cyclic codes over b-symbol read channels, IEEE Transactions on Information Theory, 62(4), 1541-1551 (2016).

Definition: Let $n_q^b(k,d)$ the minimum possible length n of an $[n,k]_q$ code with minimum distance d w.r.t. the b-symbol metric.

Linear codes in the b-symbol metric attain the Griesmer bound

Observation: Let G be a generator matrix of an $[n,k]_q$ code.

- Blocks of b subsequent columns of G span subspaces.
- The minimum distance w.r.t. the b-symbol metric equals n minus the maximum number of subspaces contained in a hyperplane.

I.e. yet another generalization of linear codes and a special subclass of additive codes.

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I.e. yet another generalization of linear codes and a special subclass of additive codes.

Griesmer type bound:

$$n_q^b(k,d) \ge \left\lceil \frac{g_q\big(k, q^{b-1}\cdot d\big)\cdot(q-1)}{q^b-1} \right\rceil = \left\lceil \frac{(q-1)\cdot \sum\limits_{i=0}^{r-1} \left\lceil d\cdot q^{b-1-i}\right\rceil}{q^b-1} \right\rceil \tag{5}$$

is attained with equality for all sufficiently large d.

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Thanks for your attention! Questions or remarks?

More research needed on additive codes and Griesmer type bounds for different settings.