Graphs of simplex codes

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September 5, 2025

Basic definitions

- $V = \mathbb{F}_q^n, \;\; q = p^m$;
- α-primitive element of $\mathbb{F}_q$;
- $[n, k]_q$-code, k-dimensional subspace of V;
- $[n]_q = \frac{q^n - 1}{q - 1}$;
- $C_i = \{(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n); x_j \in \mathbb{F}_q\}$, (Coordinate hyperplane) Kernel of the coordinate functional $c_i((x_1, x_2, \dots, x_n)) = x_i$;
- $X = \langle v_1, \dots, v_k \rangle - [n, k]_q$-code, G_x-generator matrix;

$$G_x = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$$

Basic definitions

- $C_{i,j} = C_i \cap C_j$
- Code X such that $dim(X \cap C_{i,j}) = k - 2$ for all i, j is called a projective code.
- The columns of a generator matrix of a projective code are distinct points of a projective space i.e. every two columns are linearly independent.

$$\begin{bmatrix} 1\,0\,1\,1 \\ 0\,1\,1\,2 \end{bmatrix}$$

$$\begin{bmatrix} 2\,0\,2\,2 \\ 0\,1\,1\,2 \end{bmatrix}$$

Projective codes with maximal length $n = [k]_q$ are called simplex codes.

$$\begin{bmatrix} 1\,0\,1\,1 \\ 0\,1\,1\,2 \end{bmatrix}$$

$$\begin{bmatrix} 1\,0\,1\,1 & 1 \\ 0\,1\,1\,\alpha & \alpha^2 \end{bmatrix}$$

Generator matrices of simplex codes of dimension 2 over $\mathbb{F}_3$ and $\mathbb{F}_4$.

- We say that $x \in V$ is a *simplex vector* if the Hamming weight of this vector is q^{k-1}.
- A non-zero vector is simplex if and only if it is a codeword of a certain q-ary simplex code of dimension k.
- A subspace of V is a q-ary simplex code of dimension k if and only if it is maximal with respect to the property that all non-zero vectors are simplex.

Theorem (K, Pankov, Pasini)

The simplex vectors form the algebraic variety defined by the equations

$$\sum_{i_1<\dots<i_{p^j}} x_{i_1}^{q-1}\cdots x_{i_{p^j}}^{q-1}=0$$

for $j\in\{0,\dots,mk-m-1\}$, where $q=p^m$ and p is a prime number. This variety is a quadric only if $q=2,\ k=3$ or $q=3,\ k=2$.

The group of monomial linear automorphisms of V acts transitively on the set of simplex codes and contains precisely $n!(q-1)^n$ elements. There are precisely

$$(q^k-1)(q^k-q)\dots(q^k-q^{k-1})$$

(the number of elements in $\mathrm{GL}(k,q)$) monomial linear automorphisms of V which preserve a fixed simplex code. Therefore the number of q-ary simplex codes of dimension k is equal to

$$\frac{n!(q-1)^n}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})}$$

Proposition (K, Pankov)

If X is an m-dimensional subcode of a simplex code, then every generator matrix M of X satisfies the following condition:

$(*)_m$ *M contains precisely $[k-m]_q$ zero columns and any non-zero column of M is proportional to precisely q^{k-m} columns including itself.*

If a generator matrix M of an m-dimensional code $X \subset V$ satisfies $(*)_m$, then X is a subcode of a simplex code.

$$M = \begin{bmatrix} 0&1&1&1&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1&2&2&2 \end{bmatrix}$$

M is a generator matrix of a subcode of simplex code with generator matrix

$$\begin{bmatrix} 1&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&1&1&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1&2&2&2 \end{bmatrix}$$

Theorem (K, Pankov)

Let $m \in \{0, 1, \dots, k-1\}$. Every m-dimensional subcode of a simplex code is contained in precisely

$$\frac{[k-m]_q!(q-1)^{[k-m]_q}(q^{k-m}!)^{[m]_q}}{(q^{k-m}-1)(q^{k-m}-q)\cdots(q^{k-m}-q^{k-m-1})q^{m(k-m)}}$$

distinct simplex codes. Furthermore, there are two simplex codes whose intersection is precisely this subcode except the case when $q = k = 2$.

The *Grassmann graph* $\Gamma_k(V)$ is the simple graph whose points are k-dimensional subspaces (codes) of V and two such subspaces are adjacent if their intersection is $(k-1)$-dimensional. We assume that $n = [k]_q$, $k \geq 2$ and denote by $\Gamma^s(k, q)$ the subgraph of $\Gamma_k(V)$ induced by the set of q-ary simplex codes of dimension k.
Examples:

- $\Gamma^s(2, 2)$ is a single vertex;
- $\Gamma^s(2, 3)$ is the complete bipartite graph $K_{4,4}$,
- $\Gamma^s(3, 2)$ is isomorphic to the graph $\Gamma_{1,3}(\mathbb{F}_2^4)$ formed by 1-dimensional and 3-dimensional subspaces of $\mathbb{F}_2^4$, where distinct subspaces are connected by an edge if they are incident.

A *clique* is a complete subgraph. A clique $\mathcal{X}$ is said to be *maximal* if every clique containing $\mathcal{X}$ coincides with $\mathcal{X}$. Every maximal clique of $\Gamma_k(V)$ is of one of the following type:

- the *star* $\mathcal{S}(X)$ consisting of all k-dimensional subspaces containing a certain $(k-1)$-dimensional subspace X;
- the *top* $\mathcal{T}(Y)$ consisting of all k-dimensional subspaces contained in a certain $(k+1)$-dimensional subspace Y.

The intersections of $\mathcal{S}(X)$ and $\mathcal{T}(Y)$ with the set of simplex codes are denoted by $\mathcal{S}^s(X)$ and $\mathcal{T}^s(Y)$, respectively. Every such intersection is a clique in $\Gamma^s(k,q)$ (if it is non-empty), but we cannot assert that this clique is maximal. We say that $\mathcal{S}^s(X)$ or

$\mathcal{T}^s(Y)$ is a *star* or a *top* of the simplex code graph $\Gamma^s(k,q)$ only in the case when it is a maximal clique of $\Gamma^s(k,q)$.
Since $\Gamma^s(2,3)$ and $\Gamma^s(3,2)$ are bipartite, every maximal clique in these graphs consists of two vertices which implies that it is a star and a top simultaneously. If X, Y are adjacent vertices in one of these graphs, then

$$\{X,Y\} = \mathcal{S}^s(X\cap Y) = \mathcal{T}^s(X+Y).$$

Proposition (K, Pankov)

Suppose that one of the following possibilities is realized:

- $q = 2$ *and* $k \geq 4$;
- $q = 3$ *and* $k \geq 3$;
- $q \geq 4$.

Then $\mathcal{S}^s(X)$ *is a star of* $\Gamma^s(k, q)$ *if and only if* X *is a* $(k-1)$*-dimensional subcode of a simplex code. Furthermore, there are no maximal cliques of* $\Gamma^s(k, q)$ *which are stars and tops simultaneously.*

It is clear that $\mathcal{S}^s(X)$ is non-empty if and only if X is a $(k-1)$-dimensional subcode of a simplex code. We have that

$$|\mathcal{S}^s(X)| = \frac{(q!)^{[k-1]_q}}{q^{k-1}}$$

This number is greater than $q + 1$.

If $k = 2, q \geq 4$, then $\mathcal{S}^s(X)$ consists of $(q-1)!$ elements and

$$(q-1)! \geq (q-2)(q-1) = q^2 - 3q + 2 > q + 1$$

(since $q^2 - 4q + 1 > 0$ for $q \geq 4$).
If $k \geq 3$, then

$$[k-1]_q = q^{k-2} + q^{k-3} \cdots + 1 \geq q^{k-2} + k - 2 \geq k + 1$$

(we have $q^{k-2} \geq 3$, since $k \geq 4$ if $q = 2$). Therefore,

$$\frac{(q!)^{[k-1]_q}}{q^{k-1}} \geq \frac{q^{k+1}}{q^{k-1}} = q^2 > q + 1.$$

So, $\mathcal{S}^s(X)$ contains more than $q+1$ elements and there is no $\mathcal{T}^s(Y)$ containing $\mathcal{S}^s(X)$ (since the intersection of a star and a top of the Grassmann graph is empty or contains precisely $q+1$ elements). This guarantees that $\mathcal{S}^s(X)$ is a maximal clique of $\Gamma^s(k, q)$, i.e. it is a star of $\Gamma^s(k, q)$ which is not a top.

There exist non-empty $\mathcal{T}^s(Y)$ which is not a top of $\Gamma^s(k,q)$.
Let us take $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in V$ such that

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & \alpha & \alpha^2 & \cdots & \alpha^{q-2} \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \alpha & \alpha & \alpha & \cdots & \alpha \end{bmatrix}.$$

Then $\langle \boldsymbol{x},\boldsymbol{y}\rangle$ and $\langle \boldsymbol{x},\boldsymbol{z}\rangle$ are adjacent q-ary simplex codes of dimension 2. Assume that there is a simplex code intersecting $\langle \boldsymbol{x},\boldsymbol{y}\rangle$ and $\langle \boldsymbol{x},\boldsymbol{z}\rangle$ in distinct 1-dimensional subcodes $\langle \boldsymbol{x}+\alpha^j\boldsymbol{z}\rangle, \langle \boldsymbol{x}+\alpha^i\boldsymbol{y}\rangle$, respectively. Its generator matrix is

$$\begin{bmatrix} \boldsymbol{x}+\alpha^j\boldsymbol{z} \\ \boldsymbol{x}+\alpha^i\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \alpha^j & 1 & 1+\alpha^{j+1} & \alpha+\alpha^{j+1} & \alpha^2+\alpha^{j+1} & \cdots & \alpha^{q-2}+\alpha^{j+1} \\ \alpha^i & 1 & 1+\alpha^i & \alpha+\alpha^i & \alpha^2+\alpha^i & \cdots & \alpha^{q-2}+\alpha^i \end{bmatrix}$$

and $\alpha^i \neq \alpha^j$ (since the first and second columns are non-proportional).

We choose $t \in \{0, 1, \dots, q-2\}$ such that

$$\alpha^t = \frac{\alpha^i(\alpha^{j+1} - \alpha^j)}{\alpha^j - \alpha^i}.$$

Then the determinant

$$\begin{vmatrix} \alpha^j & \alpha^t + \alpha^{j+1} \\ \alpha^i & \alpha^t + \alpha^i \end{vmatrix} = \alpha^t(\alpha^j - \alpha^i) - \alpha^i(\alpha^{j+1} - \alpha^j)$$

is zero, i.e. the first and $(t+3)$-th columns are proportional which is impossible. Therefore, every simplex code adjacent to both $\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \langle \boldsymbol{x}, \boldsymbol{z} \rangle$ belongs to the star $\mathcal{S}^s(\langle \boldsymbol{x} \rangle)$. Then $\mathcal{T}^s(\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle)$ is a non-empty proper subset of $\mathcal{S}^s(\langle \boldsymbol{x} \rangle)$ and, consequently, it is not a top.

Recall that every maximal clique of $\Gamma^s(2,3)$ and $\Gamma^s(3,2)$ is a star and a top simultaneously. Every maximal clique of $\Gamma^s(2,4)$ is a star. The graph $\Gamma^s(k,q)$ contains tops in all remaining cases.

Theorem (K, Pankov)

Suppose that one of the following possibilities is realized:

- $k = 2$ *and* $q \geq 5$,
- $k \geq 4$ *and* $q = 2$,
- $k \geq 3$ *and* $q \geq 3$.

Then $\Gamma^s(k,q)$ *contains tops. If* $k \geq 4$ *and* $q \geq 3$*, then there are tops of* $\Gamma^s(k,q)$ *containing different numbers of elements. If* $k \geq 5$ *and* $q \geq 3$*, then there is a top of* $\Gamma^s(k,q)$ *consisting of precisely three elements.*

Consider $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$ such that

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{q-3} & \alpha^{q-2} \\ 1 & \frac{1}{1+0} & \frac{1}{1+\alpha^0} & \frac{1}{1+\alpha^{-1}} & \frac{1}{1+\alpha^{-2}} & \cdots & \frac{1}{1+\alpha^{-(q-3)}} & \frac{1}{1+\alpha^{-(q-2)}} \end{bmatrix};$$

if $1 + \alpha^{-i} = 0$, then we put 0 instead of $\frac{1}{1+\alpha^{-i}}$. It is clear that the columns of $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ z \end{bmatrix}$ are mutually non-proportional, and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and $\langle \boldsymbol{x}, \boldsymbol{z} \rangle$ are simplex codes. With some calculations $\langle \boldsymbol{y}, \boldsymbol{z} \rangle$ is also a simplex code.
And $\mathcal{T}^s(\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle)$ is a top.

For $\mathbb{F}_3$ and $\mathbb{F}_4$ this vectors are linearly dependent.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

For every $i \in \{1, \dots, [k-2]_2\}$ we denote by D_i the $(k-2, 4)$-matrix whose columns are i-th non-zero vectors of $\mathbb{F}_2^{k-2}$ (in lexicografical order). and consider the $(k+1, n)$-matrix

$$M = \begin{bmatrix} A & B & \dots & B \\ \mathbf{0} & D_1 & \dots & D_{[k-2]_2} \end{bmatrix}.$$

Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ be the first, second and third rows of M (respectively). Denote by C the $(k-2)$-dimensional subspace whose generator matrix is $[\mathbf{0}, D_1, \dots, D_{[k-2]_2}]$. then $X_1 = \langle C, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle, X_2 = \langle C, \boldsymbol{v}_1, \boldsymbol{v}_3 \rangle, X_3 = \langle C, \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$ are simplex codes C is a subcode of every X_i and $X_i \cap X_j = C$
$\mathcal{T}^s(\langle X_1, X_2, X_3 \rangle)$ is a top.

$$\left[\begin{array}{c|c|c|c} 0\,1\,1 & 1\,0\,0\,1 & 1\,0\,0\,1 & 1\,0\,0\,1 \\ 1\,0\,1 & 0\,1\,0\,1 & 0\,1\,0\,1 & 0\,1\,0\,1 \\ 1\,1\,0 & 0\,0\,1\,1 & 0\,0\,1\,1 & 0\,0\,1\,1 \\ \hline 0\,0\,0 & 0\,0\,0\,0 & 1\,1\,1\,1 & 1\,1\,1\,1 \\ 0\,0\,0 & 1\,1\,1\,1 & 0\,0\,0\,0 & 1\,1\,1\,1 \end{array}\right]$$

$$\left[\begin{array}{c|c} 0\,1\,1 & 1\,0\,0\,1 \\ 1\,0\,1 & 0\,1\,0\,1 \\ 1\,1\,0 & 0\,0\,1\,1 \\ \hline 0\,0\,0 & 1\,1\,1\,1 \end{array}\right]$$

The case $q \geq 3$ and $k \geq 3$

We take $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^{q+1}$ spanning a q-ary simplex code of dimension 2. For any non-zero scalars $a, b \in \mathbb{F}$ the $(3, q+1)$-matrix

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ a\boldsymbol{x} + b\boldsymbol{y} \end{bmatrix}$$

will be denote by $A_{a,b}$. and denote by $B_{a,b}$ the $(3, q^2)$-matrix

$$[A_{a,b} \; \alpha A_{a,b} \; \dots \; \alpha^{q-2} A_{a,b} \; \mathbf{0}]$$

For every $i \in \{1, \dots, [k-2]_2\}$ we denote by D_i the $(k-2, 4)$-matrix whose columns are i-th non-zero vectors of $\mathbb{F}_q^{k-2}$ with first non-zero coordinate equal to 1 (in lexicografical order).
Let us take any collection of non-zero scalars

$$a_0, b_0, a_1, b_1, \dots, a_{[k-2]_q}, b_{[k-2]_q} \in \mathbb{F}$$

such that for some i, j we have $(a_i, b_i) \neq (a_j, b_j)$

consider the $(k+1, n)$-matrix

$$M = \begin{bmatrix} A_{a_0,b_0} & B_{a_1,b_1} & \dots & B_{a_{[k-2]_q},b_{[k-2]_q}} \\ \mathbf{0} & D_1 & \dots & D_{[k-2]_q} \end{bmatrix}.$$

Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ be the first, second and third rows of M (respectively). Denote by C the $(k-2)$-dimensional subspace whose generator matrix is $[\mathbf{0}, D_1, \dots, D_{[k-2]_2}]$. then $X_1 = \langle C, \boldsymbol{v}_2, \boldsymbol{v}_3\rangle, X_2 = \langle C, \boldsymbol{v}_1, \boldsymbol{v}_3\rangle, X_3 = \langle C, \boldsymbol{v}_1, \boldsymbol{v}_2\rangle$ are simplex codes C is a subcode of every X_i and $X_i \cap X_j = C$

Lemma

A non-zero vector of $\langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\rangle$ is not a code word of a simplex code if and only if it is a scalar multiple of

$$a_i\boldsymbol{v}_1 + b_i\boldsymbol{v}_2 - \boldsymbol{v}_3$$

for a certain $i \in \{0, 1, \dots, [k-2]_q\}$.

$\Gamma^s(2,q)$-the restriction of the Grassmann graph to the set of simplex lines.
We will use a projective terminology (1-dimensional subspaces are points and 2-dimensional are lines).

Proposition

(1) *Every simplex point is contained in precisely* $(q-1)!$ *simplex lines.*
(2) *The degree of every vertex of the graph* $\Gamma^s(2,q)$ *is equal to* $(q+1)[(q-1)!-1]$.

A simplex point $\langle 1, a_1, \dots, a_q \rangle$ is adjacent to simplex point $P = \langle 0, 1, \dots, 1 \rangle$ if and only if all columns of the matrix

$$\begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & a_1 & \dots & a_q \end{bmatrix}$$

are mutually non-proportional; the latter is equivalent to the fact that $a_1, \dots, a_q$ are mutually distinct. Therefore, there are precisely $q!$ simplex points adjacent to P.
Every simplex line passing through P contains q points distinct from P which means that P is contained in precisely $(q-1)!$ simplex lines.
Each simplex line L consists of $q+1$ simplex points. Every such point is contained in precisely $(q-1)!-1$ simplex lines distinct from L.
For $q=4, k=2$ the degree is 25

For $q = 3, 4$ the sum of $q - 1$ non-zero elements is zero if and only if these elements are mutually distinct.

For every $q \geq 5$ there are non-zero $a_1, \dots, a_{q-1} \in \mathbb{F}_q$ whose sum is zero and $a_i = a_j$ for some distinct i, j. In the case when q is odd, we take any $a \neq 0$ and $b \neq 0, a, -a$ and replace the pair $a, -a$ in the sum of all non-zero elements by the pair $b, -b$. If $q = 2^m$ with $m > 2$, then $1 + \alpha + \alpha^2 \neq 0$ and we replace the pair $1, \alpha + \alpha^2$ in the sum of all non-zero elements by the pair $1 + \alpha, \alpha^2$.

Two distinct simplex points are said to be *adjacent* if they are connected by a simplex line.

Proposition (K, Pankov)

Let $P = \langle x_1, \dots, x_{q+1} \rangle$ be a simplex point and $x_i = 0$. If a simplex point $Q = \langle y_1, \dots, y_{q+1} \rangle$ is adjacent to P, then

$$a_1 y_1 + \cdots + a_{q+1} y_{q+1} = 0, \quad \text{where} \quad a_j = x_j^{-1} \quad \text{for} \quad j \neq i \quad \text{and} \quad a_i = 0\,. \tag{1}$$

In the case when $q = 4$, a simplex point $Q = \langle y_1, \dots, y_{q+1} \rangle$ is adjacent to P if and only if it satisfies (1) and $y_i \neq 0$.

The simplex points P, Q are adjacent if and only if $y_i \neq 0$ and the columns of the matrix

$$\begin{bmatrix} x_1 & \dots & x_{q+1} \\ y_1 & \dots & y_{q+1} \end{bmatrix}$$

are mutually non-proportional. i.e. the determinants

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

are non-zero. The latter is equivalent to the fact that $y_i \neq 0$ and

$$x_1^{-1}y_1, \dots, x_{i-1}^{-1}y_{i-1},\ x_{i+1}^{-1}y_{i+1}, \dots, x_{q+1}^{-1}y_{q+1}$$

are mutually distinct elements of $\mathbb{F}_q$ (one of them is zero). The sum of all $q-1$ non-zero elements of $\mathbb{F}_q$ is zero and we obtain (1).
In the case when $q = 4$, the sum of three non-zero elements of $\mathbb{F}_q$ is zero if and only if these elements are mutually distinct. This implies the second statement.

Theorem (K, Pankov)

Let $q = 4$. Then $\Gamma^s(2,4)$ is a connected graph of diameter 3 consisting of 162 simplex lines and the degree of every vertex of Γ is equal to 25. Furthermore, for each simplex line L:

(1) *There are precisely 6 simplex lines $L_1, \dots, L_6$ which are at distance 3 from L in the graph $\Gamma^s(2,4)$.*

(2) *There are precisely 130 simplex lines which are at distance 2 from L in the graph $\Gamma^s(2,4)$. The set of all such lines is the union of three mutually disjoint subsets denoted by $\mathcal{X}^3_{20}, \mathcal{X}^1_{90}, \mathcal{X}^0_{20}$, where*
   - *$\mathcal{X}^3_{20}$ is formed by 20 simplex lines and each of these lines is adjacent to precisely three distinct L_i,*
   - *$\mathcal{X}^1_{90}$ consists of 90 lines and every such line is adjacent to a unique L_i,*
   - *$\mathcal{X}^0_{20}$ consists of 20 simplex lines disjoint with all L_i.*

(3) *$\{L, L_1, \dots, L_6\} \cup \mathcal{X}^0_{20}$ is a spread of the set of all 135 simplex points, i.e. this set consists of 27 mutually disjoint lines which cover the set of simplex points.*

For every simplex line L we denote by $\mathrm{G}(L)$ the group of all projective transformations induced by monomial semilinear automorphisms of V preserving L, i.e. the extensions of automorphisms of the corresponding simplex code (recall that every such extension is unique).

The group $\mathrm{G}(L)$ is isomorphic to $\mathrm{P\Gamma L}(2,4)$ (since the automorphism group of the corresponding code is isomorphic to $\Gamma\mathrm{L}(2,4)$)

$\mathrm{P\Gamma L}(2,4)$ is a subgroup of the permutation group S_5 acting on the points of L. These groups both are of order 120 which means that they are coincident.

Theorem (K, Pankow)

Suppose that $q = 4$ and L is a simplex line. Let $L_1, \dots, L_6$ and $\mathcal{X}^3_{20}, \mathcal{X}^1_{90}, \mathcal{X}^0_{20}$ be as in previous theorem and let $\mathcal{A}$ be the set of all simplex lines adjacent to L. The action of the group $\mathrm{G}(L)$ on the set of all simplex lines has the following properties:

(1) *The sets $\{L_1, \dots, L_6\}$, $\mathcal{X}^3_{20}$ and $\mathcal{X}^0_{20}$ are orbits of this action; moreover, the action of $\mathrm{G}(L)$ on the set $\{L_1, \dots, L_6\}$ is sharply 3-transitive.*

(2) *The set $\mathcal{A}$ is the union of two orbits consisting of 10 and 15 simplex lines.*

(3) *The set $\mathcal{X}^1_{90}$ is the union of two orbits consisting of 30 and 60 simplex lines.*

$$f(x,y) = x_1 y_1^{q-2} + x_2 y_2^{q-2} + \ldots + x_{q+1} y_{q+1}^{q-2}$$

- $f(ax + bx', y) = af(x,y) + bf(x', y)$
- $f(x, ay) = a^{q-2} f(x,y)$
- $f(y,x) = f(x^{q-2}, y^{q-2})$
- $f(x,x) = 0$ gives first equation defining simplex vectors.
- $f(x,y) = 0$ for simplex vectors x, y that give a simplex line.

for $q = 4$ we get a Hermitian form

- Geometry $S(k, q)$: point-line geometry whose maximal singular subspaces correspond to q-ary simplex codes of dimension k.
- Points: 1-dimensional subcodes of simplex codes.
- Lines: 2-dimensional subcodes of simplex codes.
- **Collinearity graph:** vertices = simplex points, edges = collinear points.

Lemma (Fisher's inequality)

if $X_1, \ldots, X_m$ are subsets of $\{1, \ldots, n\}$ such that $|X_i \cap X_j|$ is constant for all pairs of distinct i, j, then $m \leq n$.

Proposition (K,Pankov,Tyc)

Every clique of the collinearity graph of S(k, q) contains no more than n elements.

Proof

- For a clique of simplex points $\langle v_1 \rangle, \ldots, \langle v_m \rangle$, define X_i as the set of indices where the i-th coordinate of v_i is zero.
- Then $|X_i \cap X_j| = [k-2]_q$ for distinct i, j.
- By Fisher's inequality, $m \leq n$.

Definition (Power Transformation)

Let $\mathbb{F}_q$ be a finite field and s a positive integer.

- The **s-th power map** on $\mathbb{F}_q$:

$$\phi_s : a \mapsto a^s, \quad a \in \mathbb{F}_q.$$

- ϕ_s is bijective iff $\gcd(s, q-1) = 1$.
- Induces a map on $\mathbb{F}_q^n$:

$$F_s(x_1, \ldots, x_n) = (x_1^s, \ldots, x_n^s).$$

Theorem (K,Pankov,Tyc)

*If F_s is bijective and $s \neq p^m$ (not a Frobenius automorphism), then F_s sends every maximal singular subspace of $S(k, q)$ to an n-clique of the collinearity graph which is **not** a singular subspace.*

No three points of these cliques are collinear. Under additional conditions those cliques form normal rational curves.

Theorem (K,Pankov,Tyc)

Let $k = 2$, $q = p^r$, and let F_s be the s-th power map on $\mathbb{F}_q^{q+1}$:

$$F_s(x_1, \ldots, x_{q+1}) = (x_1^s, \ldots, x_{q+1}^s).$$

If the following hold:

1. $\gcd(s, q-1) = 1$ *(so F_s is bijective),*
2. $s \neq p^m$ *(not a Frobenius automorphism),*
3. ***Condition (A):*** *the p-cyclotomic coset of s modulo $q-1$ contains $up^m - 1$ for some $0 < u < p$,*

*then F_s sends every line of $S(2, q)$ to a **normal rational curve** in a projective space $\mathrm{PG}(s, q)$.*

Examples

- $q = p$, $s = p - 2$; F_{p-2} sends lines to normal rational curves in $\mathrm{PG}(p-2, p)$.
- For $k = 2$, $q = 5$: only $s = 3$ works and we have 2 types of maximal cliques: simplex lines and their images under F_3.
- For $k = 2$, $q = 7$: only $s = 5$ works but we have additional cliques that are not obtained by the power map.

Definition

A **near orthomorphism** of a group G is a bijection

$$\theta : G \setminus \{a\} \longrightarrow G \setminus \{b\}$$

such that the map

$$\delta(x) = x^{-1}\theta(x)$$

is also a bijection

$$\delta : G \setminus \{a\} \longrightarrow G \setminus \{c\}$$

for some $a, b, c \in G$.

The elements (a, b, c) are called the *ex elements* of θ.

Theorem (K)

Every top can be obtained from near-orthomorphism of the multiplicative group of the field.

Setup:

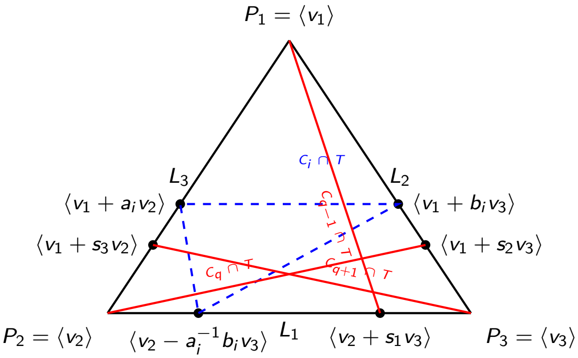
- Top T contains 3 lines $L_1 = \langle v_2, v_3 \rangle$, $L_2 = \langle v_1, v_3 \rangle$, $L_3 = \langle v_1, v_2 \rangle$.
- Lines intersect at points $P_1 = L_2 \cap L_3 = \langle v_1 \rangle$, $P_2 = L_1 \cap L_3 = \langle v_2 \rangle$, $P_3 = L_1 \cap L_2 = \langle v_3 \rangle$.

Hyperplane intersections:

- For $i \in \{1, \dots, q-2\}$:
  - $C_i \cap L_3 = \langle v_1 + a_i v_2 \rangle, \ a_i \neq a_j$ for $i \neq j$
  - $C_i \cap L_2 = \langle v_1 + b_i v_3 \rangle, \ b_i \neq b_j$ for $i \neq j$
  - $C_i \cap L_1 = \langle v_2 - b_i a_i^{-1} v_3 \rangle, \ -b_i a_i^{-1} \neq -b_j a_j^{-1}$ for $i \neq j$
- Remaining intersections:
  - $C_{q-1} \cap T = \langle v_1, v_2 + s_1 v_3 \rangle, \ s_1 \neq -b_i a_i^{-1}$
  - $C_q \cap T = \langle v_2, v_1 + s_2 v_3 \rangle, \ s_2 \neq b_i$
  - $C_{q+1} \cap T = \langle v_3, v_1 + s_3 v_2 \rangle, \ s_3 \neq a_i$

Generator Matrix for T:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\ -a_1^{-1} & -a_2^{-1} & \cdots & -a_{q-2}^{-1} & 1 & 0 & -s_3^{-1} \\ -b_1^{-1} & -b_2^{-1} & \cdots & -b_{q-2}^{-1} & -s_1^{-1} & -s_2^{-1} & 0 \end{bmatrix}$$



If we put $\theta(a_i) = -b_i$ then $\delta(a_i) = a_i^{-1}\theta(a_i) = -a_i^{-1}b_i$ and θ is a near orthomorphism of the multiplicative group of the field with ex elements (s_3, s_2, s_1).