On Intersection Families in Projective Hjelmslev Spaces

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– Finite Geometries 2025, Seventh Irsee Conference, Irsee, Germany, August 31 - September 6, 2025 –

1. Projective Hjelmslev Geometries

Theorem. Let R be a finite chain ring with radical $N = \operatorname{rad} R$. The following conditions are equivalent

(1) R is a left chain ring;

(2) the principle left ideals of R form a chain;

(3) R is local ring and $N = R\theta$ for any $\theta \in N/N^2$;

(4) R is a right chain ring.

- $\mathbb{F}_q \cong R/N$ – the residue field of R.

- m – the nilpotency index of R.

- $|R| = q^m$

- $\Gamma = \{\gamma_0 = 0, \gamma_1 = 1, \gamma_2, \ldots, \gamma_{q-1}\}$ – a set of representatives modulo N, i.e. $\gamma_i \not\equiv \gamma_j \pmod{N}$

For every $r \in R$:

$$r = r_0 + r_1\theta + \ldots + r_{m-1}\theta^{m-1}, \;\; r_i \in \Gamma,$$

where r_i are uniquely determined.

Natural homomorphism:

$$\eta_i : \begin{cases} R & \rightarrow & R/N^i \\ r_0 + r_1\theta + \ldots + r_{m-1}\theta^{m-1} & \rightarrow & (r_0 + \ldots + r_{i-1}\theta^{i-1}) + N^i \end{cases}$$

Theorem.

Let R be a finite chain ring of length m. For any finite module ${}_RM$ there exists a uniquely determined sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with

$$m \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0,$$

such that ${}_RM$ is a direct sum of cyclic modules:

$${}_RM \cong R/(\operatorname{rad} R)^{\lambda_1} \oplus R/(\operatorname{rad} R)^{\lambda_2} \oplus \ldots \oplus R/(\operatorname{rad} R)^{\lambda_k}.$$

The sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ is called the **shape** of ${}_RM$.

The sequence $\lambda' = (\lambda'_1, \dots, \lambda'_m)$, where λ'_i is the number of λ_j's with $\lambda_j \geq i$ is called the **dual shape** of ${}_RM$.

The integer k is called the **rank** of ${}_RM$.

The integer λ'_m is called the **free rank** of ${}_RM$.

The sequence $\lambda = (\underbrace{m, \cdots, m}_{a_m}, \underbrace{m-1, \cdots, m-1}_{a_{m-1}}, \cdots, \underbrace{1, \cdots, 1}_{a_1})$ is written as $m^{a_m}(m-1)^{a_{m-1}} \cdots 1^{a_1}$

Theorem.

Let R be a chain ring of length m with residue field of order q. Let ${}_RM$ be an R-module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. For every sequence $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \ldots \geq \mu_n \geq 0$, satisfying $\mu \leq \lambda$ (i.e. $\mu_i \leq \lambda_i$ for all i) the module ${}_RM$ has exactly

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^m} = \prod_{i=1}^{m} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q$$

submodules of shape μ. Here

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)}.$$

are the Gaussian coefficients.

- $M = {}_R R^n$;
- $\mathcal{P}$ – all free submodules of M of rank 1;
- $\mathcal{L}$ – all free submodules of M of rank 2;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\bigcirc_i$ - **neighbour relation**: $X \bigcirc_i Y$ iff $\eta_i(X) = \eta_i(Y)$
- $[U]^{(i)} = \{X \in \mathcal{P} \mid \exists Y \in U, X \bigcirc_i Y\}$, U is a subspace
- Hjelmslev subspaces of dimension k – free submodules of rank $k+1$;
- subspaces of shape λ – submodules of shape λ;
- Notation: $\mathrm{PHG}({}_R R^n)$, or $\mathrm{PHG}(n-1, R)$.

Let $\Sigma = (\mathcal{P}, \mathcal{L}, I) = \mathrm{PHG}(n-1, R)$.

Fix a Hjelmslev subspace S of (projective) dimension $s-1$.

Set

$$\mathfrak{P} = \{T \cap [P]^{(m-i)} \mid T \mathbin{\circ}_i S, T \cap [P]^{(m-i)} \neq \varnothing\}.$$

$\mathcal{L}(S)$ the set of all lines from $\mathcal{L}$ that are contained as a set of points in some Hjelmslev subspace T with $T \mathbin{\circ}_i S$.

$$\mathfrak{I} \subset \mathfrak{P} \times \mathcal{L}(S) \text{ by } (T \cap [P]^{(m-i)}, L) \in \mathfrak{I} \text{ iff } T \cap [P]^{(m-i)} \cap L \neq \varnothing.$$

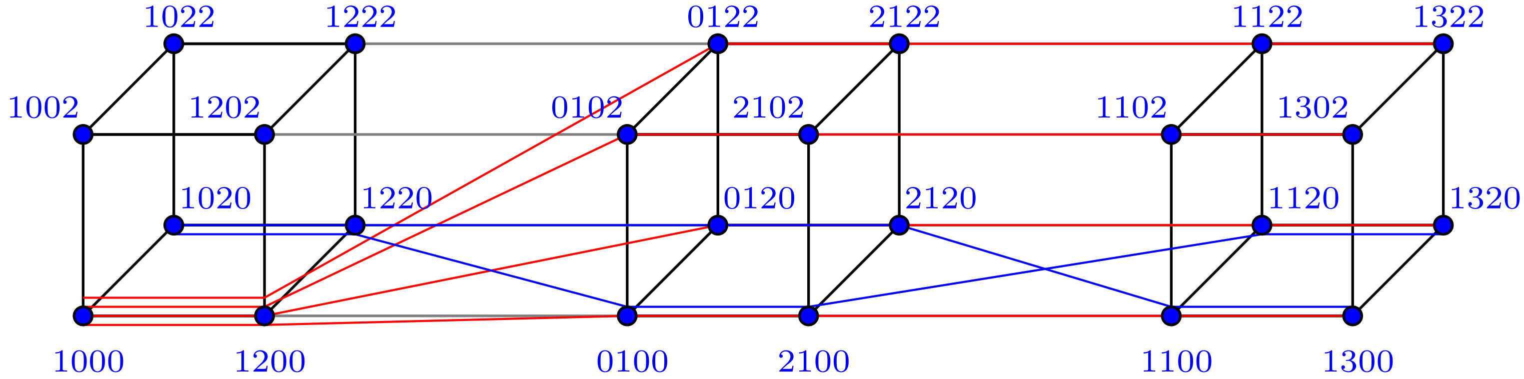
$\mathfrak{L}$ a maximal family of lines from $\mathcal{L}(S)$ that are different as subsets of $\mathfrak{P}$.

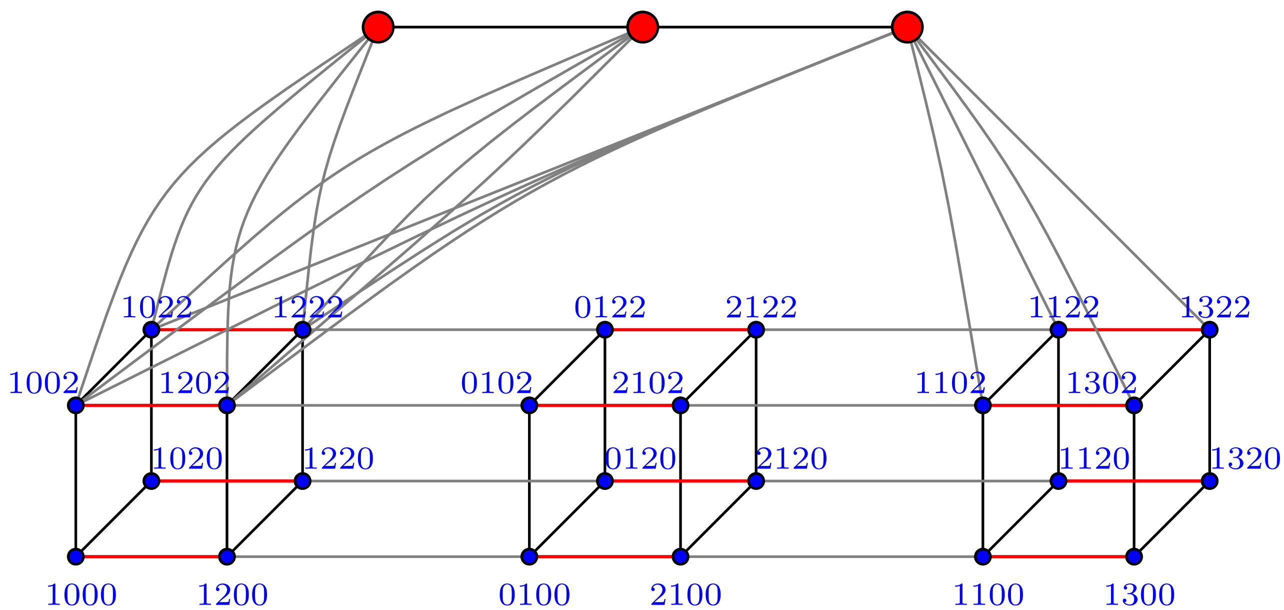
Theorem. The incidence structure $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I})$ can be imbedded isomorphically into $\mathrm{PHG}(n-1, R/N^{m-i})$. The missing part is a subspace of shape

$$(m-i)^{n-s}(m-i-1)^s$$

(i.e. a neighbour class $[U]^{(1)}$ where U is a Hjelmslev subspace of the projective geometry $\mathrm{PHG}(n-1, R/N^{m-i})$ of dimension $n-s-1$).

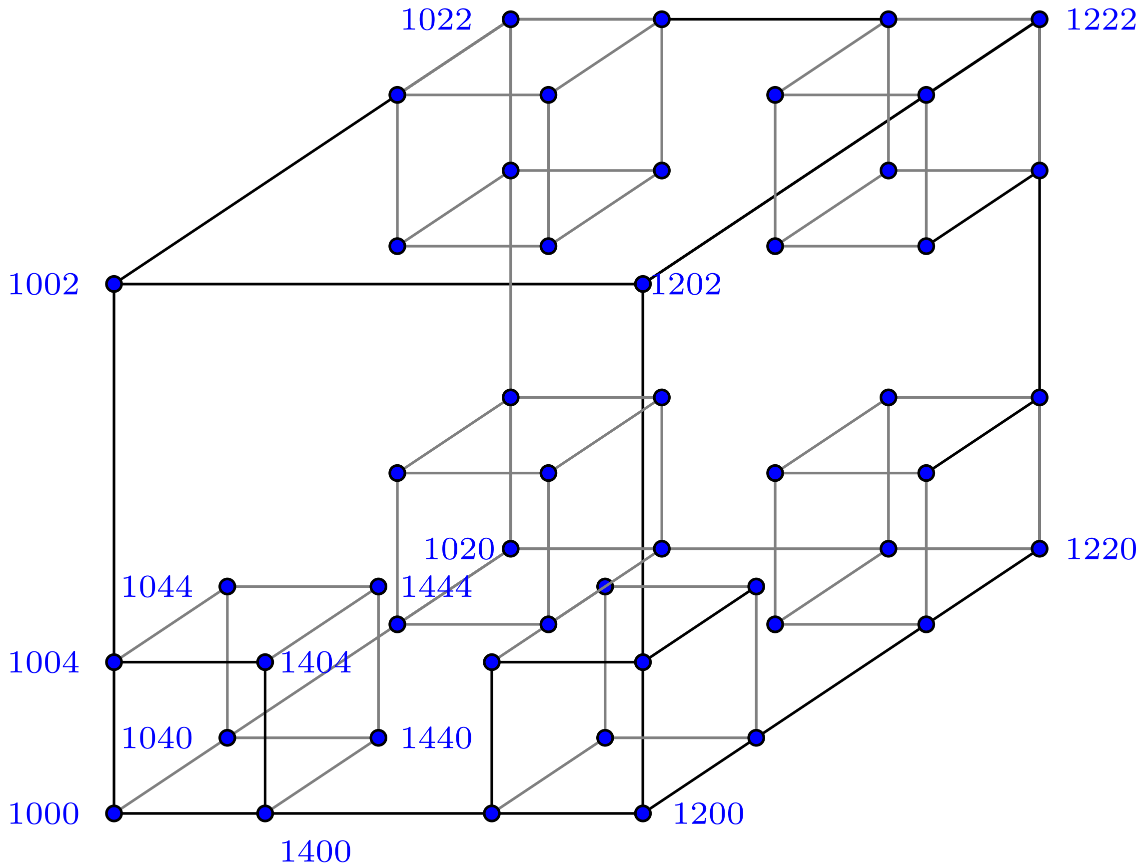
A Neighbour class of lines in $\mathrm{PHG}(3, \mathbb{Z}_4)$





The structure is ismorphic to $\mathrm{PG}(3,2) - \mathrm{PG}(1,q)$.

A Neighbour class of points in $\mathrm{PHG}(3, \mathbb{Z}_8)$



2. Intersection families of subspaces

Let $\Sigma = \mathrm{PHG}(n-1, R)$ be the (left) $(n-1)$-dimensional projective geometry over the chain ring R with

$$R/N \cong \mathbb{F}_q, \ \ |R| = q^m$$

Definition. A family $\mathcal{F}$ of subspaces of Σ of a given fixed shape κ is said to be τ-intersecting if the intersection of every two subspaces from $\mathcal{F}$ contains a subspace fo shape τ.

Problems:

(1) What is the maximal size of a τ-intersecting family of subspaces of shape κ in $\mathrm{PHG}(n-1, R)$?

(2) What is the structure fo a τ-intersecting family of maximal cardinality in Σ?

Theorem. (Hsieh,Frankl,Wilson,Tanaka)

Let t and k be integers with $0 \leq t \leq k$. Let $\mathcal{F}$ be a set of k-dimensional subspaces in $\mathrm{PG}(n, q)$ pairwise intersecting in at least a t-dimensional subspace.

If $n \geq 2k+1$, then $|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$.

Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional subspaces, containing a fixed t-dimensional subspace of $\mathrm{PG}(n, q)$, or in case of $n = 2k+1$, $\mathcal{F}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional subspace.

If $2k - t \leq n \leq 2k$, then $|\mathcal{F}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional subspace.

Consider $\mathrm{PG}(n-1,q)$.

Let $d \leq e$ be integers with $d+e=n$.

Fix a subspace W with $\dim W = e-1$.

Let $\mathcal{U}$ be the set of all subspaces U in $\mathrm{PG}(n-1,q)$ with $\dim U = d-1$, $U \cap W = \varnothing$.

Theorem. (Tanaka,2006)

Let $1 \leq t \leq d$ be an integer and let $\mathcal{F}$ be a family of subspaces from $\mathcal{U}$ with $\dim(U' \cap U'') \geq t-1$ for every two $U', U'' \in \mathcal{U}$. Then

$$|\mathcal{F}| \leq q^{(d-t)e}.$$

Equality holds iff

(a) $\mathcal{F}$ consists of all subspaces U through a fixed $(t-1)$-dimensional subspace T with $T \cap W = \varnothing$;

(b) in case of $e = d$, $\mathcal{F}$ is the set of all elements of $\mathcal{U}$ contained in a fixed $(2d-t-1)$-dimensional subspace V with $\dim V \cap W = d-t-1$.

3. Erdős-Ko-Rado-Type Theorems in Projective Hjelmslev Geometries

Theorem. Let $\mathcal{F} = \{F_1, \ldots, F_M\}$ be a τ-intersecting family of subspaces in Σ of shape κ, where

$$\kappa = m^{k_m}(m-1)^{k_{m-1}} \ldots 1^{k_1},\ \tau = m^{t_m}(m-1)^{t_{m-1}} \ldots 1^{t_1}.$$

Then $\eta_i(\mathcal{F}) = \{\eta_i(F_1), \ldots, \eta_i(F_M)\}$ is a τ'-intersecting family of subspaces in $\Sigma' = \mathrm{PHG}(n-1, R/N^i)$ of shape κ', where

$$\kappa' = i^{k_m}(i-1)^{k_{m-1}} \ldots 1^{k_{m-i+1}},\ \tau' = i^{t_m}(i-1)^{t_{m-1}} \ldots 1^{t_{m-i+1}}.$$

Theorem. (analogue of Tanaka's theorem) Let t, k, n be integers with $1 \le t < k \le n/2$, and let $\tau = m^t$, $\kappa = m^k$. Let further $\mathcal{F}$ be a τ-intersecting family of subspaces of shape κ in Σ with the additional property that the subspaces from $\mathcal{F}$ do have no common points with a neighbor calass $[W]$, where W is a Hjelmslew subspace with $\dim W = n - k - 1$. Then

$$|\mathcal{F}| \le q^{(k-t)(m(n-k-1)+1)}.$$

In case of equality, $\mathcal{F}$ is one of the following:

(a) the set of all subspaces through a fixed $(t-1)$-dimensional (Hjlemslev) subspace (U) with $U \cap [W] = \varnothing$;

(b) in the case $k = n/2$, $\mathcal{F}$ can also be the set of all $(k-1)$-dimensional subspaces on a fixed $(2k-t-1)$-dimensional subspace U with $\dim U \cap W = k-t-1$.

Theorem. Let t, k, n be integers with $1 \leq t < k \leq n/2$, and let $\tau = m^t$, $\kappa = m^k$. Let $\mathcal{F}$ be a τ-intersecting family of κ-subspaces in $\Sigma = \mathrm{PHG}(n{-}1, R)$. Then

$$|\mathcal{F}| \leq \begin{bmatrix} m^{n-t} \\ m^{k-t} \end{bmatrix}_{q^m} = q^{(m-1)(k-t)(n-k)} \begin{bmatrix} n-t \\ 1 \end{bmatrix}_q .$$

In case of equality, $\mathcal{F}$ is one of the following:

(a) all Hjelmslev subspaces of dimension $k-1$ through a fixed Hjelmslev subspace of dimension $t-1$;

(b) in the case $k \leq n/2$, $\mathcal{F}$ can also be the family of all Hjelmslev subspaces in Σ of dimension $k-1$ through a fixed subspace of dimension $n-t-1$.

4. Families of intersecting non-free subspaces

Example.

R -chain ring : $R \cong \mathbb{F}_q, |R| = q^2$

$n = 4$: $\mathrm{PHG}(3, R)$

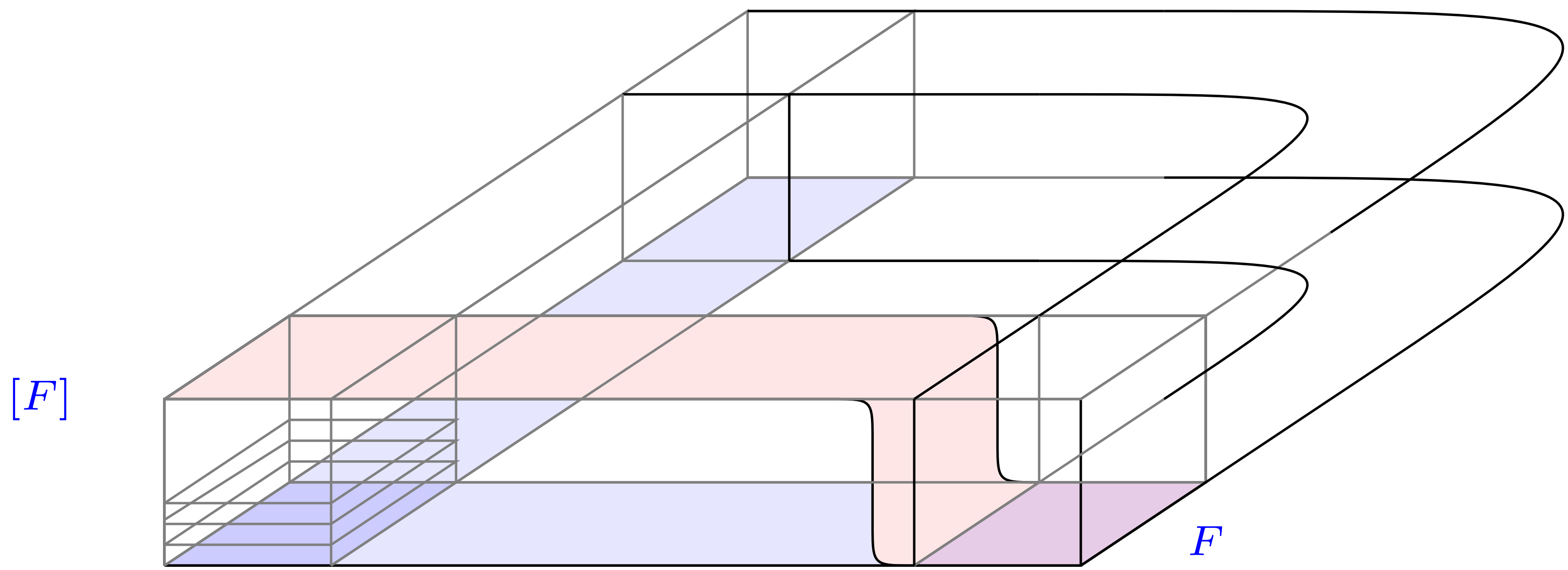
$\kappa = 2^2 1^1$ (line stripes), $\tau = 2^1$ (points)

$\mathcal{F}$ – a maximal τ-intersecting family of κ-subspaces

$\eta(\mathcal{F})$: (a) a pencil of lines, or

(b) all lines in a fixed plane of $\mathrm{PG}(3, q)$

$$|\mathcal{F}| \leq q^2(q+1)(q^2+q+1)$$


[F]
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$n_F = \#$ of neighbor classes of planes through $[F]$ containing subspaces from $\mathcal{F}$ that are not entirely contained in $[F]$.

Then $\quad |\mathcal{F} \cap [F]| \leq q^2(q+1-n_F) + qn_F.$

For $n_F = 1$: $\quad |\mathcal{F} \cap [F]| \leq q^3 + q.$

$$|\mathcal{F}| \leq (q^2+q+1)(q^3+q).$$

In fact, for a maximal family we have:

$$|\mathcal{F}| = (q^2+q+1)(q^3+1).$$

Theorem. Let R be finite chain ring with $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Let $k \geq 1$ be an integer and let $\tau = 2^1$, $\kappa = 2^k 1^{k-1}$, and $n = 2k$. Let $\mathcal{F}$ be a τ-intersecting family of κ-subspaces in $\Sigma - \mathrm{PHG}(2k-1, R)$. Then

$$|\mathcal{F}| \leq \left(q^{k+1} \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q + 1 \right) \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}_q .$$

In case of equality, $\mathcal{F}$ is the following:

all subspaces of shape κ contained in $[F]$, where F is a hyperplane in Σ, apart from those that have the "direction" of F, plus all κ-subspaces contained in F.