Intersection Density of Permutation Groups

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Permutations

Definition

Two permutations σ and π in $\mathrm{Sym}(n)$ **agree or intersect** if for some $i \in \{1, 2, \ldots, n\}$

$$i^{\sigma} = i^{\pi}, \text{or, equivalently, } i^{\sigma\pi^{-1}} = i$$

- A permutation is a **derangement** if it has no fixed points.
- Permutations σ and π are intersecting if and only if $\pi^{-1}\sigma$ is **not** a derangement.

Definition

A set of permutations is an **intersecting set** if any pair elements from the set are intersecting.

What is the largest set of intersecting permutations in a permutation group?

This depends on the group action!

Intersection Density

Definition

For a transitive group G, the set

$$S_{i,j} = \{\sigma \in G \,|\, i^\sigma = j\}.$$

is a **canonical intersecting set**. $G_i = S_{i,i}$ is the stabilizer of a point.

Lemma

Any transitive group G with degree n has an intersecting set of size $|G_x| = \frac{|G|}{n}$.

We will only consider transitive permutation groups.

Definition (Li, Song, Tej Pantangi)

The **intersection density of a transitive group** G is

$$\rho(G) := \max\left\{\frac{|\mathcal{F}|}{|G_x|} \,|\, \mathcal{F} \subseteq G \text{ is an intersecting set}\right\}.$$

If the intersection density for a transitive permutation group is equal to 1, then the group has the **Erdős-Ko-Rado or EKR property**.

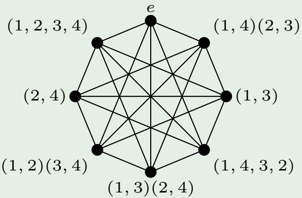
Definition

For any permutation group G we define a **derangement graph**, Γ_G.

- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\sigma\pi^{-1}$ is a derangement.

Permutations are adjacent if they are **not** intersecting.

Example



The graph $\Gamma_{D(4)}$.

Properties of the Derangement Graphs

An intersecting set in G is a coclique (independent set / stable set) in Γ_G.

The graph Γ_G is really nice:

- The derangement graph is **regular**, all the vertices have the same number of neighbours.
- The **degree** equal the number of derangements in G.
- G is a subgroup of the automorphism group of Γ_G.
- This graph is **vertex transitive**; the automorphism group acts transitively on the vertices,

Graph Homomorphisms - a Reduction

Theorem

If $H \leq G$ are permutations groups and H is transitive, then $\rho(G) \leq \rho(H)$.

Proof.

There is a graph homomorphism (embedding) $\Gamma_H \hookrightarrow \Gamma_G$. Homomorphism bounds fractional chromatic number, so

$$\frac{|H|}{\alpha(\Gamma_H)} = \chi_f(\Gamma_H) \leq \chi_f(\Gamma_G) = \frac{|G|}{\alpha(\Gamma_G)}$$

which implies

$$\rho(G) = \frac{n\,\alpha(\Gamma_G)}{|G|} \leq \frac{n\,\alpha(\Gamma_H)}{|H|} = \rho(H).$$

Corollary

If $H \leq G$ with H transitive and H has EKR property, then G has the EKR property.

Regular subgroups

Theorem

If G has a regular subgroup then $\rho(G) = 1$.

Proof.

Let H be the regular (sharply transitive) subgroup. If n is the degree of G, then $|H| = n$ and no elements of H, except the identity, fix a point. For any distinct $x, y \in H$, this implies that xy^{-1} is not a derangement, so x and y are adjacent in Γ_H. Γ_H is a n-clique, and $\rho(H) = \frac{\alpha(\Gamma_H)}{\frac{|H|}{n}} = 1$.

The following groups have the EKR property:

- The symmetric group [Frankl and Deza, 1977],
- any group with an n-cycle, and
- all Frobenius groups.
- Transitive groups with prime degree p (and with degree p^k).

Eigenvalues of Cayley Graphs

1. The derangement graph is a **normal** Cayley graph

$$\Gamma_G = \mathrm{Cay}(G, \mathrm{Der}(G))$$

2. vertices are elements of G; vertices σ, π are adjacent if $\sigma\pi^{-1} \in \mathrm{Der}(G)$
3. The **connection set** of the Cayley graph is set of derangements; so it is closed under conjugation.

If C_i is a conjugacy class of G, then we can define

$$\Gamma_{G,i} = \mathrm{Cay}(G, C_i), \quad \text{so} \quad \Gamma_G = \bigcup_{\substack{C_i \text{ a class of} \\ \text{derangements}}} \mathrm{Cay}(G, C_i)$$

The adjacency matrix of the derangement graph is

$$A(\Gamma_G) = \sum_i A(\mathrm{Cay}(G, C_i)).$$

Further, the matrices $A(\mathrm{Cay}(G, C_i))$ commute and are simultaneously diagonalizable.

Eigenvalues

Theorem

The eigenvalues of $\Gamma_{G,i} = \mathrm{Cay}(G, C_i)$ *are*

$$\xi_\chi = \frac{\chi(\sigma)|C_i|}{\chi(1)}$$

where χ *is an irreducible character of* G*, and* $\sigma \in C_i$*.*

Theorem

The eigenvalues of Γ_G *are*

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_i \chi(\sigma)|C_i|$$

where the χ *is an irreducible representation of* G *and the sum is over all conjugacy classes of derangements in* G*.*

Unions and Joins

Lemma

If $Der(G)$ is the set of derangements of a group G, then Γ_G is connected if and only if $\langle Der(G) \rangle = G$.

- The multiplicity of the degree equals the number of components.

Lemma

Let H be the subgroup generated by the elements of G that are **not** *derangements. If H is a proper subgroup of G then Γ_G is the join of graphs Γ_H.*

- Γ_G is formed by taking copies of Γ_H, and adding edges between any two vertices in different copies of Γ_H
- If the difference of the degree and the smallest eigenvalue is equal to the number of vertices, then Γ_G is a join.

Example (Gap's TransitiveGroup (8,13))

Structure Description is $C2 \times A4$, its size is 24.

The spectrum is: $\{15^{(1)}, \quad 3^{(4)}, \quad -1^{(18)}, \quad -9^{(1)}\}$
(Since $15 - (-9) = 24$, so the graph is a join.)
The complement has spectrum: $\{8^{(2)}, \quad 0^{(18)}, \quad -4^{(4)}\}$
(This is the disjoint union of two graphs with 12 vertices.)
The complement of one of these graphs has spectrum: $\{3^{(3)}, -1^{(9)}\}$
(This is the disjoint union of three copies of K_4.)

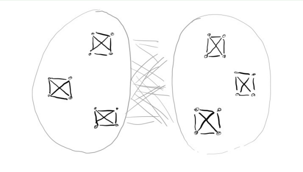


Figure: Artist's rendering of $\Gamma_{C_2 \times A_4}$.

This graph is a **cograph**, or **complement-reducible graph**.

Delsarte / Hoffman Ratio Bound

For $M = \sum_i x_i A(\mathrm{Cay}(G, C_i))$ a weighted adjacency matrix for Γ_G,

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{d}{\tau}}$$

where d is the row sum and τ is the least eigenvalue for M.

The eigenvalues of $M = \sum_i x_i A(\mathrm{Cay}(G, C_i))$, are

$$\frac{1}{\chi(1)} \sum_i x_i \chi(\sigma) |C_i|.$$

(This is an easy way to make weighted adjacency matrices for which we can still calculated eigenvalues.)

Eigenvalues

To get the best bound in the ratio bound we set this up as a linear programming problem:

$$\text{Maximize}: \quad d = \sum_i x_i |C_i|,$$

$$\text{Subject to}$$

$$-1 \leq \frac{1}{\chi_1(1)} \sum_i x_i \; \chi_1(\sigma)|C_i|$$

$$\cdots$$

$$-1 \leq \frac{1}{\chi_\ell(1)} \sum_i x_i \; \chi_\ell(\sigma)|C_i|$$

This is very easy to solve for specific groups (I have been using Gurobi) or for families of groups for which the characters are well-understood.

Open Problems

Cody Antal, Cody Solie and I have a database of intersection density of small groups. The data base is at https://intersection-density.vercel.app/

Filter Reset Filters Hide/Show Columns

Degree	Gap ID	Order	Structure Description	Stabilizer Description	Upper Bound	Lower Bound	Intersection Density	Transitivity
18	753	69984	((C3 x C3 x C3 x C3 x C3 x C3) : ((C4 x C2) : C4)) : C3	((C3 x C3 x C3 x C3) : (C4 x C2)) x S3	2	2	2	1
18	473	5184	(C2 x C2 x C2 x C2 x C2 x C2) : ((C3 x C3 x C3) : C3)	C2 x A4 x A4	2	2	2	1
18	198	648	(S3 x S3 x S3) : C3	S3 x S3	1.0277778	1.0277778	1.0277778	1
18	607	17496	(((C3 x C3) : C2) x ((C3 x C3) : C2) x ((C3 x C3) : C2)) : C3	((C3 x C3 x C3) : C2) x ((C3 x C3) : C2)	1.0277778	1.0277778	1.0277778	1
18	315	1296	((S3 x S3 x S3) : C3) : C2	C2 x S3 x S3	1.5	1.5	1.5	1
18	3	18	C3 x S3	1	1	1	1	1
18	4	18	(C3 x C3) : C2	1	1	1	1	1
18	5	18	D9	1	1	1	1	1
18	6	36	C6 x S3	C2	1	1	1	1
18	11	36	S3 x S3	C2	1	1	1	1
18	10	36	(C3 x C3) : C4	C2	1	1	1	1
18	9	36	S3 x S3	C2	1	1	1	1
18	12	36	C2 x ((C3 x C3) : C2)	C2	1	1	1	1
18	13	36	D18	C2	1	1	1	1
18	14	54	C2 x (C9 : C3)	C3	1	1	1	1
18	16	54	C9 x S3	C3	1	1	1	1

We have the intersection density for all transitive permutation groups with degree less than or equal to 19. We are working on degree 20-23, but think 24 is out of reach.