On the flag-transitive automorphism groups of 2-designs with λ prime

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2-designs

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Definition

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- A **flag** is any incident point-block pair of $\mathcal{D}$.

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- G acts **flag-transitively** on $\mathcal{D}$ if for any flags (x, B) and (x', B') of $\mathcal{D}$ there is $\gamma \in G$ such that $(x^\gamma, B^\gamma) = (x', B')$.

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flag-transitivity $\Rightarrow$block-transitivity $\Rightarrow$point-transitivity

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We are interested in the case where G acts flag-transitively on $\mathcal{D}$.

Flag-transitivity & Point-primitivity

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A point-transitive automorphism group G of $\mathcal{D}$ is said to be **point-imprimitive** if G preserves a partition Σ of the point-set of $\mathcal{D}$ in classes of size v_0 with $1 < v_0 < v$.

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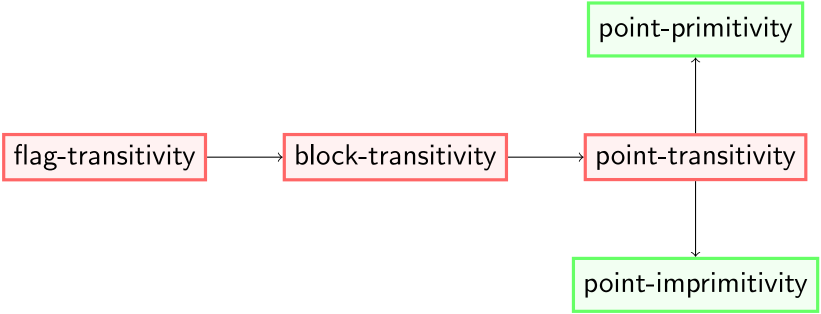
$$\boxed{\text{flag-transitivity}} \longrightarrow \boxed{\text{block-transitivity}} \longrightarrow \boxed{\text{point-transitivity}}$$

Flag-transitivity & Point-primitivity



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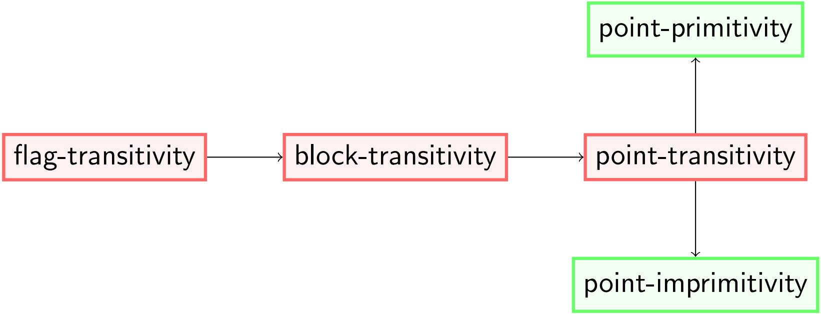

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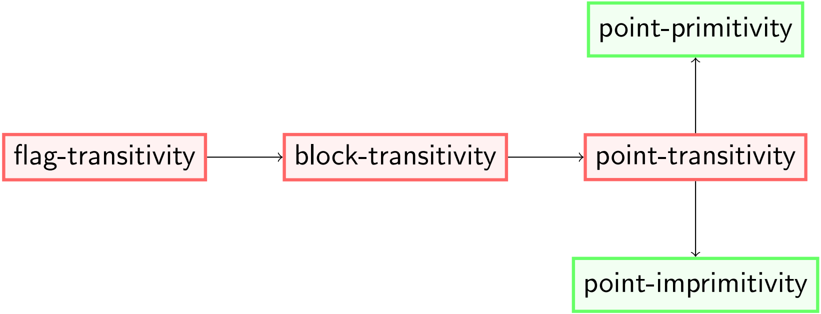

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Theorem (Higman-McLaughlin, 1961)

Any flag-transitive automorphism group of a 2-design with $\lambda = 1$ acts point-primitively.

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- $\mathcal{D}$ is one of the two $(16, 6, 2)$ biplanes with G isomorphic to $(Z_2)^4 : S_4$ or $(Z_2 \times Z_8).(S_4.Z_2)$, respectively, (Husain (1945) and, independently, by Nandi (1946), and O'Relly-Reguerio (2005));

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Theorem (Davies, 1987)

For any fixed λ, there are only finitely many 2-(v, k, λ) designs with a flag-transitive point-imprimitive automorphism group.

Conditions ensuring point-primitivity

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Theorem

Let G be any flag-transitive automorphism group of a 2-(v, k, λ) design $\mathcal{D}$. Then G acts point-primitively on $\mathcal{D}$, provided that at least one of the following conditions on the parameters of $\mathcal{D}$ holds:

Line	Condition	Author(s)
1	$\lambda > (r, \lambda) \cdot ((r, \lambda) - 1)$	Dembowski, 1968, or
2	$(r, \lambda) = 1$	Kantor, 1969
3	$(r - \lambda, k) = 1$	
4	$r > \lambda(k - 3)$	
5	$(v - 1, k - 1) = 1$ or 2	
6	$k > 2\lambda^2(\lambda - 1)$	Devillers-Praeger, 2021–2023
7	$v > \left(2\lambda^2(\lambda - 1) - 2\right)^2$	
8	$\lambda \leq 4$ and except for eleven specific $\mathcal{D}$	
9	$(v - 1, k - 1)^2 \leq v - 1$	Zhong-Zhou, 2023
10	$(v - 1, k - 1) = 3$ or 4	
11	k prime	

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Theorem (M., 2025)

Let G be any flag-transitive point-imprimitive automorphism group of a 2-(v, k, λ) design $\mathcal{D}$. If λ is a prime, then one of the following holds:

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3. $\mathcal{D}$ is a 2-$(2^{2^{j+1}}(2^{2^j}+2), 2^{2^j}(2^{2^j}+1), 2^{2^j}+1)$ design when $2^{2^j}+1 > 3$ is a Fermat prime.

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There are no known examples corresponding to case (3).

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- match the results obtained on $(\mathcal{D}_0, G_\Delta^\Delta)$ and on $(\mathcal{D}_1, G^\Sigma)$.

Indice

Flag-transitive point-primitive 2-designs with small λ

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Theorem

Let $\mathcal{D}$ be a non-trivial 2-(v, k, λ) design admitting a flag-transitive point-primitive automorphism group G. If $G \not\leq A\Gamma L_1(v)$, v power of a prime, then $(\mathcal{D}, G)$ is classified in the following cases:

Conditions on $\mathcal{D}$	Conditions on G	Author(s)
$\lambda = 1$		Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck Saxl, 1990
$\lambda = 2, v = b$		O'Relly-Reguerio, 2005
$\lambda = 2, v < b$	G almost simple	Alavi, Devillers, Daneshkah, Liang, M., Praeger, Xia, Zhou et. al 2016–2025
$\lambda = 2, v < b$	G affine	Liang-M., 2025
$2 < \lambda \leq 10, v = b$	G affine	Alavi-Daneshkhah-M., 2025+

Example 2 (from $SL_n(q)$ or $\mathcal{C}_j \cup \mathcal{S}$-subgroups, $j = 3, 8$)

Let $V = V_n(q)$, where $q = p^{d/n}$, and let $G = T : G_0$, $x \in V^*$, $\mathbb{F}_q^* = \langle \omega \rangle$ and $\sigma : (y_1, ..., y_n) \mapsto (y_1^p, ..., y_n^p)$. Then the following hold:

(v, k, r, b, λ)	Base Block	G_0	$Aut(\mathcal{D})$
$\left(p^d, 3, p^d - 1, \frac{p^d(p^d-1)}{3}, 2\right)$	$\left\langle \omega^{\frac{(p^d-1)j}{3}} \right\rangle x$	$SL_n(q) \trianglelefteq G_0$ $Sp_n(q) \trianglelefteq G_0$ $G_2(q) \trianglelefteq G_0$ $n = 6$, q even $GL_1(q^n) \trianglelefteq G_0$	$A\Gamma L_n(q)$
$(p^d, p^t, 2\frac{p^d-1}{p^t-1}, 2p^{d-t}\frac{p^d-1}{p^t-1}, 2)$	$\langle x \rangle_{GF(p^t)}$	$GL_n(q) : \left\langle \sigma^{t/2} \right\rangle$	G

Note that, $p^d \equiv 1 \pmod 3$ in the first family of examples, t is a proper even divisor of d/n in the second one.

Example 3 (from $\mathcal{C}_6$-subgroups)

Line	(v, k, r, b, λ)	Base Block	G_0
1	$(5^2, 4, 16, 100, 2)$	$\{(0,0), (0,1), (\omega, \omega^3), (\omega^3, \omega^3)\}$	$(Z_4 \times Z_4) : Z_2$
2	$(7^2, 3, 48, 784, 2)$	$\{(0,0), (0,1), (1, \omega)\}$	$Z_3 \times Z_2.S_4^-$
3			$Z_2.S_4^-$
4			$Z_3 \times Q_{16}$
5	$(11^2, 3, 120, 4840, 2)$	$\{(0,0), (0,1), (\omega^3, \omega^4)\}$	$Z_5 \times GL_2(3)$
6		$\{(0,0), (0,1), (\omega^4, \omega^2)\}$	
7		$\{(0,0), (0,1), (\omega^2, \omega^2)\}$	$Z_5 \times SL_2(3)$
8	$(11^2, 4, 80, 2420, 2)$	$\{(0,0), (0,1), (\omega^4, \omega), (\omega^9, \omega^5)\}$	$Z_5 \times SD_{16}$
9	$(19^2, 6, 144, 8664, 2)$	$\{(0,0), (0,1), (\omega^4, \omega^5), (\omega^4, \omega^{14}), (\omega^7, \omega), (\omega^7, \omega^{17})\}$	$Z_9 \times GL_2(3)$
10			$Z_9 \times SD_{16}$
11		$\{(0,0), (0,1), (\omega^5, \omega^{11}), (\omega^5, \omega^{13}), (\omega^8, \omega^{10}), (\omega^8, \omega^{14})\}$	$Z_9 \times GL_2(3)$
12			$Z_9 \times SD_{16}$
13		$\{(0,0), (0,1), (1, \omega^{12}), (1, \omega^{13}), (\omega^{15}, \omega^9), (\omega^{15}, \omega^{14})\}$	$Z_9 \times SD_{16}$
14		$\{(0,0), (0,1), (\omega, \omega^{10}), (\omega, \omega^{17}), (\omega^4, \omega^2), (\omega^4, \omega^{11})\}$	
15		$\{(0,0), (0,1), (\omega^2, \omega^2), (\omega^2, \omega^{12}), (\omega^5, \omega^5), (\omega^5, \omega^{16})\}$	
16		$\{(0,0), (0,1), (\omega^3, \omega^9), (\omega^3, \omega^{12}), (\omega^6, \omega^2), (\omega^6, \omega^{15})\}$	
17	$(23^2, 3, 528, 93104, 2)$	$\{(0,0), (0,1), (\omega^6, \omega^7)\}$	$Z_{11} \times Z_2.S_4^-$
18		$\{(0,0), (0,1), (\omega^7, \omega^{17})\}$	
19		$\{(0,0), (0,1), (\omega^8, \omega^{17})\}$	
20		$\{(0,0), (0,1), (\omega^{10}, \omega^{17})\}$	

Example 4 (from $\mathcal{C}_j \cup \mathcal{S}$-subgroups, $j = 4, 6$)

Line	(v, k, r, b, λ)	Base Block	G_0
1	$(2^6, 2^2, 42, 672, 2)$	$\left\langle e_1 \otimes e_1', e_2 \otimes e_1' + e_3 \otimes e_2' \right\rangle_{\mathbb{F}_2}$	$Z_7 \times S_3$
2		$\left\langle e_1 \otimes e_1', e_2 \otimes e_1' + (e_2 + e_3) \otimes e_2' \right\rangle_{\mathbb{F}_2}$	$Z_7 \times S_3$
3	$(2^6, 7, 21, 192, 2)$	$\langle e_1, e_2, e_3 \rangle_{\mathbb{F}_2}^{*} \otimes e_1'$	Z_{21}
4			$F_{21} \times Z_3$
5			$PSL_2(7) \times Z_3$
6	$(2^6, 7, 21, 192, 2)$	$\{e_1^{\gamma^i} \otimes e_1' + e_2^{\gamma^i} \otimes e_2'\}_{i=0}^{6}$	Z_{21}
7	$(3^4, 6, 16, 216, 2)$	$\langle e_1 \rangle_{\mathbb{F}_3} \cup \left(\langle e_1 \rangle_{\mathbb{F}_3} + e_2 + e_3 \right)$	$\left(\left(Z_2 . S_4^- \right) : Z_2 \right) : Z_2$
8			$\left(Z_2 . S_4^- \right) : Z_2$
9			$((Z_8 \times Z_2) : Z_2) : Z_3$
10			$(((Z_4 \times Z_2) : Z_2) : Z_3) : Z_2$
11			$(Z_2 \times SD_{16}) : Z_2$
12			$Z_2 \times SD_{16}$
13			$(Z_8 \times Z_2) : Z_2$
14			$(Z_8 : Z_2) : Z_2$
15			$(Z_2 \times Z_2) . (Z_4 \times Z_2)$
16			$Z_4 . D_8$
17			$(Z_8 \times Z_2) : Z_2$
18			$Z_8 : (Z_2 \times Z_2)$
19			$(Z_2 \times Q_8) : Z_2$
20	$(3^4, 3^2, 20, 180, 2)$	$\langle e_1, e_2 \rangle_{\mathbb{F}_3}$	$(Z_8 \circ SL_2(5)) : Z_2$
21			$Z_8 \circ SL_2(5)$
22			$(Z_4 \circ SL_2(5)) : Z_2$ (two copies)
23			$(D_8 \circ Q_8) . F_{10}$

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Theorem (Zhang-Chen-Zhou, 2024)

Let $\mathcal{D}$ be a nontrivial 2-(v,k,λ) design with λ prime admitting a flag-transitive and point-primitive automorphism group G with socle $T \cong \mathrm{A}_n$, $n \geq 5$. Then one of the following holds:

1. $\mathcal{D}$ is a 2-$(6,3,2)$ design and $G \cong A_5$;
2. $\mathcal{D}$ is a 2-$(10,4,2)$ design and $G \cong A_5, S_5, A_6, P\Sigma L_2(9)$;
3. $\mathcal{D}$ is a 2-$(10,6,5)$ design and $G \cong A_5, S_5, A_6, S_6$;
4. $\mathcal{D}$ is a 2-$(15,7,3)$ design and $G \cong A_7, A_8$.

Sporadic Groups

Theorem (Alavi-Daneshkhah-M., 2025)

Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design with λ prime admitting a flag-transitive and point-primitive automorphism group G with socle T a simple sporadic group. Then $(\mathcal{D}, G)$ is (up to isomorphism) as one of the rows in the following table.

Table: Sporadic simple groups and flag-transitive 2-designs with λ prime.

Line	v	b	r	k	λ	G	G_α	G_B
1	12	22	11	6	5	M_{11}	$\mathrm{PSL}_2(11)$	A_6
2	22	77	21	6	5	M_{22}	$\mathrm{PSU}_3(4)$	$2^4{:}\mathrm{A}_6$
	22	77	21	6	5	$\mathrm{M}_{22}{:}2$	$\mathrm{PSU}_3(4){:}2$	$2^4{:}\mathrm{S}_6$
3	176	1100	50	8	2	HS	$\mathrm{PSU}_3(5){:}2$	S_8

Exceptional Lie Type Groups

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1. T is ${}^2\mathrm{B}_2(\mathrm{q})$ with $q-1>3$ is a Mersenne prime, and $\mathcal{D}$ is the 2-$(q^2+1, q, q-1)$ design arising from the Suzuki-Tits ovoid;

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Example 5

Using the Higman-McLaughlin setting: $\cos(T, H, K) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where

- $\mathcal{P} = \{Hx : x \in T\}$, $\mathcal{B} = \{Ky : y \in T\}$;
- $Hx \,\mathcal{I}\, Ky$ if and only if $Hx \cap Ky \neq \varnothing$.

The Affine Case

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Theorem (Alavi-Bayat-Daneshkhah-M., 2025)

Let $\mathcal{D}$ be a nontrivial symmetric design with λ prime admitting a flag-transitive and point-primitive automorphism group G of affine type. Then $G \leq A\Gamma L_1(q)$, or $\mathcal{D}$ is a symmetric 2-$(16, 6, 2)$ design with full automorphism group $2^4 : \mathrm{S}_6$ and point-stabilizer S_6.

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Example 6 (Buratti-Martinović-Nakić, 2025)

There are two non isomorphic flag-transitive 2-$(3^3, 6, 5)$ designs with $AGL_1(3^3) \trianglelefteq G \leq A\Gamma L_1(3^3)$.

THANK YOU FOR YOUR ATTENTION!