On the Q-polynomial property of bipartite graphs with a uniform structure

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(Joint work with B. Fernández, R. Maleki, and Š. Miklavič)

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Notations and preliminaries

$\Gamma = (X, \mathcal{R})$: (simple, connected, undirected) graph with vertex set X and edge set $\mathcal{R}$

$\partial(x, y)$: distance between $x, y \in X$

$\varepsilon(x) = \max\{\partial(x, y) \mid y \in X\}$: *eccentricity* of x

$D = \max\{\varepsilon(x) \mid x \in X\}$: *diameter* of Γ

$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ $(\Gamma(x) := \Gamma_1(x))$

V : $\mathbb{R}$-vector space of column vectors indexed by X

M : $\mathbb{R}$-algebra of matrices with rows and columns indexed by X

The adjacency matrix

$A \in M$ with

$$(A)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = 1, \\ 0 & \text{otherwise} \end{cases}$$

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An A^*-type matrix

$x \in X$, $\varepsilon := \varepsilon(x)$
$\{\theta^*_i\}_{i=0}^{\varepsilon}$: pairwise distinct (real) scalars

$A^* := A^*(\theta^*_0, \theta^*_1, \ldots, \theta^*_\varepsilon; x) \in M$ is diagonal with

$$(A^*)_{yy} = \theta^*_i \iff \partial(x,y) = i \qquad (y \in X).$$

A generalized Q-polynomial property

Let $\theta_0 > \theta_1 > \ldots > \theta_{\mathcal{D}}$ be the eigenvalues of A and $V_i \, (0 \leq i \leq \mathcal{D})$ the ith eigenspace of A.

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Γ is *Q-polynomial* *w.r.t.* x if

(i) there is an A^*-type matrix (w.r.t. x),

(ii) there is an ordering $V_{i_0}, V_{i_1}, \ldots, V_{i_{\mathcal{D}}}$ of the eigenspaces of A such that

$$A^* V_{i_j} \subseteq V_{i_{j-1}} + V_{i_j} + V_{i_{j+1}} \qquad (0 \leq j \leq \mathcal{D}).$$

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$$A^* V_{i_j} \subseteq V_{i_{j-1}} + V_{i_j} + V_{i_{j+1}} \qquad (0 \leq j \leq \mathcal{D}).$$

$\Rightarrow$ A^* is a *dual adjacency matrix of Γ w.r.t. x and $\{V_{i_j}\}_{i=0}^{\mathcal{D}}$.*

Where does it come from? What's *new*?

Fact: A distance-regular graph which is Q-polynomial in the *standard* sense has a dual adjacency matrix for every $x \in X$.

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(i) extends to any (simple, connected, undirected) graph,

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Purpose: find examples of graphs which are Q-polynomial in the *new* sense.

A dual adjacency matrix candidate

Often an A^*-type matrix is a dual adjacency matrix of Γ if A and A^* satisfy a *tridiagonal relation*:

$$A^3A^* - A^*A^3 + (\beta+1)(AA^*A^2 - A^2A^*A) = \\ = \gamma(A^2A^* - A^*A^2) + \rho(AA^* - A^*A)$$

for $\beta, \gamma, \rho \in \mathbb{R}$.

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for $\beta, \gamma, \rho \in \mathbb{R}$.

$\Rightarrow$ A^* is a *dual adjacency matrix candidate* *w. r. t.* x if there are $\beta, \gamma, \rho \in \mathbb{R}$ such that a tridiagonal relation is satisfied.

$\Rightarrow$ β, γ, ρ are the *corresponding parameters* of the dual adjacency matrix candidate.

Finding a dual adjacency matrix (candidate)

Assumption: Γ is bipartite and *uniform* w. r. t. x (definition is coming soon).

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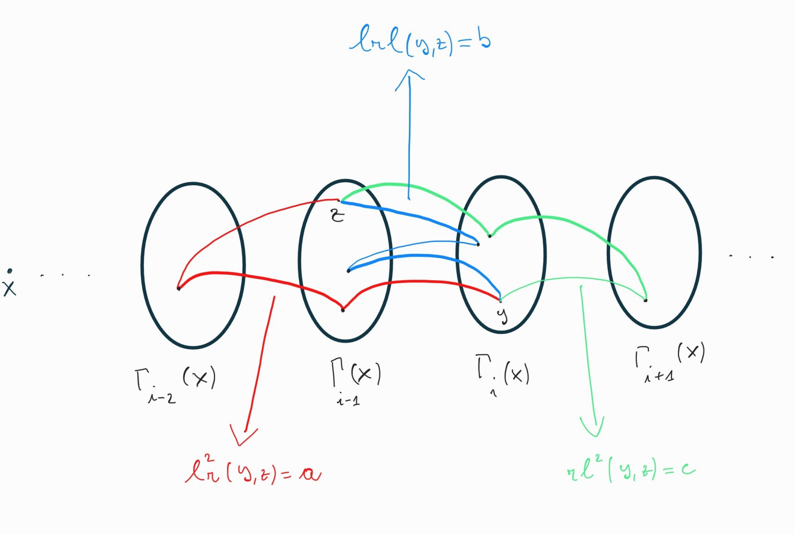
Our project:

(i) provide sufficient conditions on the *uniform structure* of Γ such that Γ has a dual adjacency matrix candidate w.r.t. x;

(ii) produce graphs which are Q-polynomial in the sense of the new definition using (i).

On the graph in question: Γ bipartite and uniform

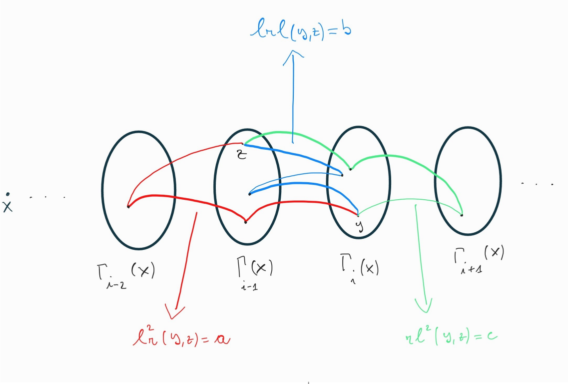
If Γ is bipartite, there are three possible 3-walk shapes from y to z:



On the graph in question: Γ bipartite and uniform

Γ (bipartite) is *uniform* *w.r.t.* x if there are $e_i^-, e_i^+, f_i \in \mathbb{R}$ such that

$$e_i^- a + b + e_i^+ c = \begin{cases} f_i & \text{if } \partial(z,y) = 1, \\ 0 & \text{otherwise} \end{cases} \qquad (1 \le i \le \varepsilon).$$



(+ some *technical conditions* on e_i^-, e_i^+, f_i)

Theorem (restrictions on e_i^+, e_i^-, f_i, and θ_i^*)

Let Γ be a uniform (bipartite) graph and $A^* = A^*(\theta_0^*, \ldots, \theta_\varepsilon^*)$ $(\theta_i^* \neq \theta_j^*,\ i \neq j)$. If there are scalars $\beta, \rho \in \mathbb{R}$ such that

$$\beta + 1 = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (2 \leq i \leq \varepsilon - 1),$$

$$e_i^-(\beta + 2) = 1 + (\beta + 1)\frac{\theta_{i-1}^* - \theta_{i-2}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (2 \leq i \leq \varepsilon),$$

$$e_i^+(\beta + 2) = 1 + (\beta + 1)\frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-1}^*} \qquad (1 \leq i \leq \varepsilon - 1),$$

$$f_i(\beta + 2) = \rho \qquad (1 \leq i \leq \varepsilon);$$

then A^* is a dual adjacency matrix candidate (w.r.t. x) with corresponding parameters β, ρ.

Applications of *our algorithm*

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Full bipartite graph

Assume that $G = (X, \mathcal{R})$ is a non-bipartite graph, and $x \in X$. Consider the graph

$$G_f := G_f(x) = (X, \mathcal{R}_f),$$

where

$$\mathcal{R}_f = \mathcal{R} \setminus \{yz \mid \partial(x,y) = \partial(x,z)\}.$$

$\Rightarrow$ G_f (connected and bipartite) is called the *full bipartite graph of G w.r.t. x*.

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Theorem (B. Fernández, R. Maleki, G.M., Š. Miklavič, 2023)

If G is either a *Hamming* or a *dual polar graph*, then G_f is uniform (w.r.t. x).

Hamming graph

The full bipartite graph of the Hamming graph $H(D, n)$, $n \neq 2$, is uniform w.r.t. x with coefficients

$$e_i^- = -\frac{1}{2} \quad (2 \leq i \leq D), \qquad e_i^+ = -\frac{1}{2} \quad (1 \leq i \leq D-1),$$

$$f_i = n - 1 \quad (1 \leq i \leq D).$$

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Theorem 2

Let $\Gamma = H(D,n)_f$, $n \neq 2$, (w.r.t. x). Pick $\theta_0^*, \theta_1^* \in \mathbb{R}$ such that $\theta_0^* \neq \theta_1^*$, and set

$$\theta_{i+1}^* = \theta_1^* + (\theta_1^* - \theta_0^*)i \quad (1 \leq i \leq D-1).$$

Then $A^* = A^*(\theta_0^*, \theta_1^*, \ldots, \theta_D^*)$ is a dual adjacency matrix candidate of Γ w.r.t. x, with $\beta = 2$, $\rho = 4(n-1)$.

Dual polar graph

The full bipartite graph of the dual polar graph $\Delta(D, b, e)$, $e \neq 0$, is uniform w.r.t. x with coefficients

$$e_i^- = -\frac{b^2}{b+1} \quad (2 \leq i \leq D), \qquad e_i^+ = -\frac{b^{-1}}{b+1} \quad (1 \leq i \leq D-1),$$

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$$\theta_{i+1}^* = \theta_1^* + (\theta_1^* - \theta_0^*)\frac{b^i - 1}{b^i(b-1)} \quad (1 \leq i \leq D-1).$$

Then $A^* = A^*(\theta_0^*, \theta_1^*, \ldots, \theta_D^*)$ is a dual adjacency matrix candidate of Γ w.r.t. x, with $\beta = b + b^{-1}$, $\rho = b^{e+D-1}(b + b^{-1} + 2)$.

The main result

Main Theorem

Let Γ be either $H(D,n)_f$ or $\Delta(D,b,e)_f$ (w.r.t. x).
Then, there is a dual adjacency matrix of Γ w.r.t. x and the following orderings of eigenspaces of A:

(i) $V_0 < V_2 < \cdots < V_{2D} < V_1 < V_3 < \cdots < V_{2D-1}$;

(ii) $V_1 < V_3 < \cdots < V_{2D-1} < V_0 < V_2 < \cdots < V_{2D}$.

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Corollary

Let Γ be either $H(D,n)_f$ or $\Delta(D,b,e)_f$ (w.r.t. x).
Then Γ is Q-polynomial w.r.t. x.

Thank you for your attention!

Appendix A: Why a tridiagonal relation is the *key*

Let $\theta_0 > \theta_1 > \ldots > \theta_{\mathcal{D}}$ be the eigenvalues of A and E_i $(0 \leq i \leq \mathcal{D})$ the corresponding *primitive idempotent*.
Assume A^* is a dual adjacency matrix candidate.

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Then, for some $\beta, \gamma, \rho \in \mathbb{R}$,

$$E_i(A^3A^* - A^*A^3)E_j + (\beta+1)E_i(AA^*A^2 - A^2A^*A)E_j =$$
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that is,

$$(\theta_i - \theta_j)\left(\theta_i^2 + \theta_j^2 - \beta\theta_i\theta_j - \gamma(\theta_i + \theta_j) - \rho\right) E_iA^*E_j = 0$$

since $E_iA = \theta_iE_i = AE_i$ $(0 \le i \le \mathcal{D})$.

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$\Rightarrow$ What are the i, j such that $E_i A^* E_j = 0$?

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$\Rightarrow$ What are the i, j such that $E_i A^* E_j = 0$?

The *best* (for our aim):

$$E_i A^* E_j = 0 \quad \text{for all } i, j \text{ with } i - j \notin \{-a, 0, a\} \quad (a \in \mathbb{N}).$$

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then

$$A^* V_j \subseteq V_{j-a} + V_j + V_{j+a} \qquad (0 \leq j \leq \mathcal{D})$$

since $E_i V = V_i$ and $\sum_{i=0}^{\mathcal{D}} E_i = I$.

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$\Rightarrow$ A^* is a dual adjacency matrix, and Γ is Q-polynomial (w.r.t. x).

Appendix B: On the graph in question (Γ bipartite and uniform)

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$\Rightarrow$ For such a graph the *Terwilliger algebra* (w.r.t. x), say T, has some very useful, interesting properties:

(i) *every irreducible T-module, say W, is thin;*

(ii) *the isomorphism class of W only depends on the endpoint and diameter of W.*

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$\Rightarrow$ This helps us to understand

- the algebraic structure (eigenvalues),
- the combinatorial properties (3-walks)

of a *potentially* Q-polynomial graph.

Appendix C: More on dual polar graphs

Let U be a finite-dimensional vector space over $\mathbb{F}_b$ (b prime power) with one of the nondegenerate forms below:

name	dim U	form	e
$C_D(b)$	$2D$	symplectic	1
$B_D(b)$	$2D+1$	quadratic	1
$D_D(b)$	$2D$	quadratic (Witt index D)	0
${}^2D_{D+1}(b)$	$2D+2$	quadratic (Witt index D)	2
${}^2A_{2D}(q)$	$2D+1$	Hermitean $\left(b=q^2\right)$	$\frac{3}{2}$
${}^2A_{2D-1}(q)$	$2D$	Hermitean $\left(b=q^2\right)$	$\frac{1}{2}$

The *dual polar graph* $\Delta(D,b,e)$ has as points the maximal isotropic subspaces (dimension D), and two points x, y are adjacent if $dim(x \cap y) = D-1$.

Appendix D: From tridiagonal relation to Q-polynomial property

If Γ is either $H(D,n)_f$ or $\Delta(D,b,e)_f$, there are $\beta,\rho\in\mathbb{R}$ such that

$$A^3A^* - A^*A^3 + (\beta+1)(AA^*A^2 - A^2A^*A) = \rho(AA^* - A^*A),$$

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Proposition 2

$$E_i A^* E_j = 0 \text{ for } 0 \leq i, j \leq 2D \text{ such that } |i - j| \notin \{0, 2\}.$$

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(V_i is the ith eigenspace of A and E_i is the ith primitive idempotent)