On sets of points of $\mathrm{PG}(n, q)$ with few intersection numbers

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Outline

1. A combinatorial problem in finite geometries
2. Type of a set of $\mathrm{PG}(n, q)$
3. Recent results on the parameters of a set of hyperplane–type $(m, h)_{n-1}$
4. A combinatorial characterization of the the complement of a hyperbolic quadric

A combinatorial problem in finite projective spaces

- *A set of points of* $\mathrm{PG}(2, q)$*, q odd, no three of which are collinear has size* $k \leq q + 1$ *and equality occurs iff it is a (irreducible) conic* (B. Segre, 1954)

- *A set of points of* $\mathrm{PG}(3, q)$*, q odd, no three of which are collinear has size* $k \leq q^2 + 1$ *with equality iff it is an elliptic quadric* (A. Barlotti, 1955; G. Panella, 1955)

- *A set* $\mathcal{K}$ *of points of* $\mathrm{PG}(n, q)$ *intersected by any hyperplane in 1 or h points either is a line or* $n = 3$ *and* $\mathcal{K}$ *is an ovoid* (J. Thas, 1973)

Study k–sets of points of $\mathrm{PG}(n, q)$ *with respect to their intersection with one (or more than one) prescribed family of subspaces of* $\mathrm{PG}(d, q)$*:*

– Admissibility of parameters and Examples

– Classification and characterization results

A characterization problem: B.Segre point of view in finite geometries

Geometric and algebraic properties of certain structures may be be derived from seemingly superficial and very few data:

- *In the classes of arcs and caps conics and elliptic quadrics are characterized by their sizes.*
- *Elliptic quadrics are characterized as sets of non–collinear points of* $\mathrm{PG}(3, q)$ *via their intersections with respect to hyperplanes.*

B. Segre point of view: Characterize classical stuctures by their combinatorial properties. (A. Beutelspacher, 1988).

The type of a k–set in $\mathrm{PG}(n, q)$

Let $\mathbb{P} = \mathrm{PG}(n, q)$ and $m_1, m_2, \cdots, m_s$ be s integers such that $0 \leq m_1 < m_2 < \cdots < m_s \leq q + 1$.

A subset $\mathcal{K}$ of points of $\mathbb{P}$ is of type $(m_1, m_2, \ldots, m_s)_h$ with respect to the family $\mathcal{P}_h$ of all h–dimensional subspaces of $\mathbb{P}$ if $|H \cap \mathcal{K}| \in \{m_1, m_2, \ldots, m_s\}$ for every $H \in \mathcal{P}_h$ and any m_j (intersection number) , ($j = 1, \ldots, h$), occurs as the size of intersection of $\mathcal{K}$ with a member of $\mathcal{P}_h$.

If $h = 1$ or $h = n - 1$ $\mathcal{K}$ is of *line–type* $(m_1, m_2, \ldots, m_s)_1$ and of *hyperplane–type* $(m_1, m_2, \ldots, m_s)_{n-1}$, respectively.

In $\mathrm{PG}(2, q)$ (non—degenerate) conics are of line–type $(0, 1, 2)_1$ and in $\mathrm{PG}(3, q)$ elliptic quadrics are of line type $(0, 1, 2)_1$ and of plane–type $(1, q+1)_2$.

$c_j^h :=$ the number of h–dimensional subspaces intersecting $\mathcal{K}$ in exactly j points: $\mathcal{K}$ is of type $(m_1, m_2, \dots, m_s)_h$ if $c_{m_j}^h \neq 0$ for every $j \in 1, \dots, s$ (characters of $\mathcal{K}$).

Let $\mathcal{K}$ be a set of points of $\mathrm{PG}(n, q)$, a line ℓ is external(tangent) to $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = 0$($|\ell \cap \mathcal{K}| = 1$).

Let $\mathcal{K}$ be a non–empty set of points of $\mathrm{PG}(r, q)$ (with $\mathcal{K} \neq \mathrm{PG}(r, q)$), $\mathcal{P}_h$ be the family of all the h–dimensional subspaces of $\mathrm{PG}(r, q)$ and

$$m := \min\{|H \cap \mathcal{K}|, \quad H \in \mathcal{P}_h, H \cap \mathcal{K} \neq \emptyset\}$$

$$n := \max\{|H \cap \mathcal{K}|, \quad H \in \mathcal{P}_h\}$$

$$\sum_{s=m}^{n} (n-s)(m-s) = -f_h(k, m, n) + m \cdot h \cdot c_0(h)$$

If $c_0(h) = 0$ then

$$f_h(k, m, n) \leq 0 \text{ and equality holds iff } \mathcal{K} \text{ is of type } (m, n)_h$$

The hyperplane–type case

k–sets of $\mathrm{PG}(n, q)$ of hyperplane–type are the geometric counterpart of a class of linear codes and the distribution of the intersection numbers is associated with the distribution of the weights of the corresponding code.

Sets of points of of $\mathrm{PG}(n, q)$ of hyperplane–type $(m, h)_{n-1}$ are associated with strongly regular graphs and (l, m)–difference sets.

If $\mathcal{K}$ is a set of hyperplane–type $(m, h)_{n-1}$ then $h - m \mid q^{n-1}$ (so $h \leq m + q^{n-1}$). (Tallini Scafati 1976)

Sets of $\mathrm{PG}(3, q)$ of plane type $(m, h)_2$

$$h \leq m + q^2$$

$$\mathcal{K} \text{ a plane} \Rightarrow h = m + q^2 = (q+1) + q^2$$

Some classical objects, such as e.g. Hyperbolic quadrics and Hermitian surfaces, satisfy $h = m + q$.

If $m \leq q$ then $h \leq m + q$.

If there are both an external line and a tangent line then $h \leq m + q$.

For $n = 3$ the result of J. Thas shows that $h = m + q = 1 + q$

A set of points of $\mathrm{PG}(3, q)$ *of plane–type* $(2, h)_2$ *points is the union of two skew lines, and so* $h = m + q = 2 + q$*.*
(N.Durante–D.Olanda 2006)

- *If $\mathcal{K}$ is a set of points of $\mathrm{PG}(3,q)$, $q > 2$, of plane–type $(3,h)_2$ then $h \leq q+3$, and if equality occurs then if $q > 4$ $\mathcal{K}$ is the union of three skew lines. If $q = 4$ $\mathcal{K}$ is either the union of three skew lines or $\mathrm{PG}(3,2)$ embedded in $\mathrm{PG}(3,4)$. If $q = 3$ then $\mathcal{K}$ is the union of three skew lines or $k \in \{12,15\}$ and there are three examples of such sets of plane type $(3,6)_2$ (of which one with $k = 12$).* (V.N.–D.Olanda, 2012)

- *If $h < q+3$ for $q = 8$ there is an example of a set of plane–type $(3,7)$.*

For $q = 2$ K is either a plane, or the whole space $\mathrm{PG}(3,2)$ or the set of points on three pairwise skew lines.

If $\mathcal{K}$ is a set of points of $\mathrm{PG}(3,q)$ of plane–type $(3,h)_2$, then

- *$\mathcal{K}$ is the set of the points of a plane of $\mathrm{PG}(3,2)$*
- *$h = q + 3$*
- *$q = 8$ and $h = 7$.* (F. Zuanni, 2023)

In $\mathrm{PG}(3,q)$, apart from the planes of PG(3,3), for sets of plane–type $(4,h)_2$ we have $h = 4 + q$. (S.Innamorati, 2024)

The general case

The planar case:

$\mathcal{K}$ a set of points of a finite projective plane of order q of line–type $(m, n)_1$ then

If $(m, n)_1 = 1 = (m-1, n-1)_1$, $m \geq 2$, *then either* $n - m < \sqrt{q}$ *or* q *is a square,* $n - m = \sqrt{q}$ *and* $k = m(q + \sqrt{q} + 1)$ *or* $k = q\sqrt{q} + \sqrt{q}(\sqrt{q} - 1)(m - 1) + m$. [G. Tallini, J. Geom. (1987)]

The general case

Higher dimensions:

Let K be a k-set of points of $\mathrm{PG}(r, q)$ of hyperplane-type $(m, n)_{r-1}$, $r \geq 2$, $q = p^h$ and $h \geq 1$. Assume $n - m > q^{\frac{r-1}{2}}$. Then either $m \equiv n \equiv k \equiv 0 \mod p$ or $m \equiv n \equiv k \equiv 1 \mod p$.
[V.N. Austral. J. Combin. 2022]

Thus, if K is a k–set of hyperplane-type $(m, n)_{r-1}$, $r \geq 2$, then either $n - m \leq q^{\frac{r-1}{2}}$ or p divides m and n or p divides $m - 1$ and $n - 1$, where p is the prime number such that $q = p^h$ and $h \geq 1$.

Variations and generalizations of the characterization problem

• Extra geometric and/or combinatorial conditions: e.g. intersection sizes with all the members of another family of subspaces, conditions on some sets of subspaces, existence of special sets of subspaces,...

• Extra algebraic conditions: e.g. being an algebraic (hyper)surface of a precribed order.

• Characterizations of a family of subspaces of $\mathrm{PG}(n, q)$ which behaves as a family of subspaces of the space with respect to a classical object of $\mathrm{PG}(n, q)$ and so reconstructions of classical objects.

Theorem (B. Sahu, Austral. J. Combin. (2022))

Let Σ be a non–empty family of planes of $\mathrm{PG}(3, q)$, for which the following properties are satisfied:
(P1) Every point of $\mathrm{PG}(3, q)$ is contained in $q^2 - q$ or q^2 planes of Σ.
(P2) Every line of $\mathrm{PG}(3, q)$ is contained in 0, $q - 1$, q or $q + 1$ planes of Σ.
Then Σ is the set of all planes of $\mathrm{PG}(3, q)$ meeting a hyperbolic quadric in an irreducible conic.

Theorem (V.N., submitted)

Let q be a prime power and m be a positive integer with $m \leq q$. Assume that $\mathcal{K}$ is a set of points of $\mathrm{PG}(3, q)$ intersected by any plane in $q^2 - m$ or q^2 points such that there is at least one external line to $\mathcal{K}$. Then, $\mathcal{K}$ is of plane–type $(q^2 - m, q^2)_2$, $m = q$, $(q+1)$–lines, q–lines and $(q-1)$–lines do exist and if $\mathcal{K}$ is of line–type $(0, q-1, q, q+1)_1$ then it is the complement of the set of points of a hyperbolic quadric of $\mathrm{PG}(3, q)$.