The second minimum size of a subspace partition

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- $V = V(n,q)$ the vector space of dimension n over $\mathbb{F}_q$.
- A subspace partition or partition $\mathcal{P}$ of V, is a collection of subspaces $\{W_1, \ldots, W_k\}$ s.t.
  - $V = W_1 \cup \cdots \cup W_k$
  - $W_i \cap W_j = \{\mathbf{0}\}$ for $i \neq j$.
- size of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$.

General Applications

Subspace Partitions are used to

- build block designs
- build error-correcting codes
- viewed as subspace codes
- decompose graphs into subgraphs

General Problem

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Lemma (Herzog, Schönheim, 1972; Beutelspacher 1975; Bu 1980).
Let d be an integer such that $1 \leq d \leq n/2$. Then $V(n, q)$ admits a partition with

- 1 subspace of dim $n - d$, and
- q^{n-d} subspaces of dim d.

Question: What is the minimum number of subspaces in a partition?

Beutelspacher Lemma (1980) The minimum possible size $m_q(n)$ over all (nontrivial) subspace partition of $V(n,q)$ for $n \geq 2$ is

$$m_q(n) = q^{\lceil n/2 \rceil} + 1.$$

- $\sigma_q(n,t) =$ the min size of any partition of $V(n,q)$ in which the largest subspace has dim t.

- if $t|n$ then André (1954), Segre (1964)

$$\sigma_q(n,t) = \frac{q^n - 1}{q^t - 1}.$$

Let $\ell = \dfrac{q^{n-t} - q^r}{q^t - 1}$.

Theorem (Heden, Lehmann, N., and Sissokho, 2011, 2012). Let $\mathcal{P}$ be a partition of $V(n, q)$ with largest subspace dim t and $r \equiv n \pmod{t}$. Then

the *min* $|\mathcal{P}|$ is

$$\sigma_q(n,t) = \begin{cases} q^t + 1, & \text{if } r \geq 1 \text{ and } 3 \leq n < 2t, \\ \ell q^t + q^{\lceil \frac{t+r}{2} \rceil} + 1, & \text{if } r \geq 1 \text{ and } n \geq 2t. \end{cases}$$

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Question: What types of partitions are of size $\sigma_q(n, t)$?

Theorem (Heden, Lehmann, N., and Sissokho, 2014). Let $n > t > 0$ be integers with $r \equiv n \pmod{t}$ and $1 \leq r < t$. If $\mathcal{P}$ is a partition of $V(n,q)$ of size $\sigma_q(n,t)$, then $\mathcal{P}$ has type:

- $s^{q^s+1} t^{\ell q^t}$, if $t + r = 2s$
- $(t-1)^{q^t} t^{\ell q^t + 1}$ if $t + r = 2t - 1$
- $(s-1)^{q^s} s^1 t^{\ell q^t}$ if $r < t - 1$ and $t + r = 2s - 1$

Open Question: If $n_t < \ell q^t$ and $t + r = 2s - 1$ are there partitions of size $\sigma_q(n,t)$?

Beutelspacher Lemma (1980). The min $|\mathcal{P}|$ over all (nontrivial) partitions $\mathcal{P}$ of $V(n,q)$ for $n \geq 2$ is

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Theorem (Extension of Beutelspacher's Lemma).
Let $m'_q(n) = min\ |\mathcal{P}|$ over all (nontrivial) partitions $\mathcal{P}$ of $V(n,q)$ with $|\mathcal{P}| > m_q(n)$. Then,

$$m'_q(n) = \begin{cases} \dfrac{q^n - 1}{q-1} = m_q(n), & \text{if } n \in \{1,2\} \\ q^{\lceil n/2 \rceil} + q^{\lceil (n-1)/4 \rceil} + 1 = m_q(n) + q^{\lceil (n-1)/4 \rceil}, & \text{if } n \geq 3. \end{cases}$$

- $\sigma'_q(n,t) =$ second min size of any partition $\mathcal{P}$ of $V(n,q)$ in which the largest subspace has dim t, i.e any $\mathcal{P}$ with $|\mathcal{P}| > \sigma_q(n,t)$.

- $\sigma'_q(n,t)$ = second min size of any partition $\mathcal{P}$ of $V(n,q)$ in which the largest subspace has dim t, i.e any $\mathcal{P}$ with $|\mathcal{P}| > \sigma_q(n,t)$.
- Recall $\ell = \dfrac{q^{n-t} - q^r}{q^t - 1}$.

Theorem. Let $\mathcal{P}$ be a partition of $V(n,q)$ with largest subspace dim t, second largest dim s, and $|\mathcal{P}| > \sigma_q(n,t)$. If $t + r \neq 2s - 1$, where $r \equiv n \pmod{t}$, and $n_t \leq \ell q^t + 1$, then

Proof Sketch: First,

$$\sigma'_q(n,t) \geq \begin{cases} \ell q^t + q^{\lceil t/2 \rceil} + 1, & \text{if } r = 0; \\ q^t + q^{\lceil r/2 \rceil} + 1, & \text{if } r \geq 1 \text{ and } n < 2t; \\ \ell q^t + q^{\lceil (t+r)/2 \rceil} + q^{\lceil (t+r-1)/4 \rceil} + 1, & \text{if } r \geq 1 \text{ and } n \geq 2t. \end{cases}$$

Proof Sketch (cont.): Let $\mathcal{P}$ be a partition of $V(n,q)$ that is:

- $r = 0$.
  - a t-spread: replace a dim t space X with $\mathcal{P}_X$ with $|\mathcal{P}_X| = m_q(t)$. Then obtain $\mathcal{P}^*$ with

$$|\mathcal{P}^*| = |\mathcal{P}| - 1 + |\mathcal{P}_X| = \sigma(d,t) - 1 + m_q(t) = \ell q^t + q^{\lceil t/2 \rceil} + 1.$$

- $r \geq 1$ and $n = t + r < 2t$.
  - type $t^1 r^{q^r}$: replace a dim r space Y with $\mathcal{P}_Y$ with $|\mathcal{P}_Y| = m_q(r)$. Then obtain $\mathcal{P}^*$ with

$$|\mathcal{P}^*| = |\mathcal{P}| - 1 + |\mathcal{P}_Y| = \sigma(d,t) - 1 + m_q(r) = q^t + q^{\lceil r/2 \rceil} + 1.$$

- $r \geq 1$ and $n > 2t$.
  - type $(t+r)^1 t^{\ell q^t}$: replace dim $t+r$ space Z with $\mathcal{P}_Z$ with $|\mathcal{P}_Z| = m'_q(t+r)$. Then obtain $\mathcal{P}^*$ with

$$|\mathcal{P}^*| = |\mathcal{P}| - 1 + |\mathcal{P}_Z| = \ell q^t + m'_q(t+r) = \ell q^t + q^{\lceil \frac{t+r}{2} \rceil} + q^{\lceil \frac{t+r-1}{4} \rceil} + 1.$$

Theorem. Let $\mathcal{P}$ be a partition of $V(n,q)$ with largest subspace dim t, second largest dim s, and $|\mathcal{P}| > \sigma_q(n,t)$. If $t+r \neq 2s-1$, where $r \equiv n \pmod{t}$, and $n_t \leq \ell q^t + 1$, then $|\mathcal{P}| \geq \sigma'_q(n,t)$, with

$$\sigma'_q(n,t) = \begin{cases} \sigma_q(n,t) + q^{\lceil t/2 \rceil} = \sigma_q(d,t) + m_q(t) - 1 & \text{if } r = 0; \\ \sigma_q(n,t) + q^{\lceil r/2 \rceil} = \sigma_q(d,t) + m_q(r) - 1, & \text{if } r \geq 1 \text{ and } n < 2t; \\ \sigma_q(n,t) + q^{\lceil (t+r-1)/4 \rceil} = \sigma_q(d,t) - m_q(t+r) + m'_q(t+r), & \text{if } r \geq 1 \text{ and } n \geq 2t. \end{cases}$$

Let $\mathcal{P}$ be a partition of $V(n,q)$ of type $d_1^{n_{d_1}} \dots d_k^{n_{d_k}}$.

- The t-supertail of $\mathcal{P}$, with $d_1 < t \leq d_k$, is defined to be the set $\mathcal{S}$ $= \{W \in \mathcal{P} : \dim W < t\}$, and set $s = \max_{W \in \mathcal{S}} \dim W$.

Theorem (Heden, Lehmann, N., and Sissokho, 2013). Let $\mathcal{P}$ be a partition of $V(n, q)$. If $\mathcal{S}$ is a t-supertail of $\mathcal{P}$, then

$$|\mathcal{S}| \geq \sigma_q(t, s).$$

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Question. If $\mathcal{P}$ is a partition of $V(n, q)$ with t-supertail

$$|\mathcal{S}| > \sigma_q(s, t),$$

then what is the minimum size of $\mathcal{S}$?

Theorem. Let $\mathcal{P}$ be a partition of $V(n,q)$ with largest subspace dim t, second largest subspace dim $s>1$, $r \equiv n \pmod{t}$, $u \equiv t \pmod{s}$, $s+u \neq 2u-1$, and let $\mathcal{S}$ be the *t-supertail*.

If $|\mathcal{S}| > \sigma_q(t,s)$, then $|\mathcal{S}| \geq \sigma'_q(t,s)$, where

$$\sigma'_q(t,s) = \begin{cases} \dfrac{q^t-q^s}{q^s-1} + q^{\lceil s/2 \rceil} + 1, & \text{if } u = 0, \\ \dfrac{q^t-q^{s+u}}{q^s-1} + q^{\lceil (s+u)/2 \rceil} + q^{\lceil (s+u-1)/4 \rceil} + 1, & \text{if } u \geq 1 \text{ and } t \geq 2s \\ q^s + q^{\lceil s/2 \rceil} + q^{\lceil (s-1)/4 \rceil} + 1, & \text{if } u \geq 1 \text{ and } 2s - r < t < 2s \end{cases}$$

- A partial spread is a subset of subspaces of $V(n, q)$ of dimension t. It is called maximal if cannot be extended to a larger one.

Corollary. Let $\mathcal{T}$ be a maximal partial t-spread of $V(n, q)$, $n > 3t$, $0 \leq r < t - 1$, and $t + r \neq 2s - 1$. If $|\mathcal{T}| > \sigma_q(n - t + 1, t)$, then

$$|\mathcal{T}| \geq \sigma'_q(n - t + 1, t).$$

Conjecture. Let $\mathcal{P}$ be a partition of $V(n,q)$ with largest subspace dim t, second largest subspace dim s, $r \equiv n \pmod{t}$, and having at least three different subspace dimensions. If $|\mathcal{P}| > \sigma_q(n,t)$, and

(a) $r < t-1$ and $t+r = 2s-1$, or

(b) $r = t-1$ and $n_t > \ell q^t + 1$,

then

$$|\mathcal{P}| \geq \sigma_q'(n,t).$$

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$$|\mathcal{P}| \geq \sigma'_q(n,t).$$

Remark. The only known partition where $n_t > \ell q^t + 1$ is when $n = 8, t = 3, r = 2$, and $q = 2$, and the conjecture holds.

Conjecture. Let $\mathcal{P}$ be a partition of $V(n,q)$ with largest subspace dim t, $r \equiv n \pmod{t}$, t-supertail $\mathcal{S}$, and suppose $|\mathcal{S}| > \sigma_q(t,s)$.
If $n_t \leq \ell q^t$ and $t + r \leq 2s - 1$ then

$$|\mathcal{S}| \geq \sigma_q'(t,s).$$

Thank you!