On line-parallelisms of $\mathrm{PG}(3, q)$

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(joint work with P. Santonastaso)

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Seventh Irsee Conference

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Kirkman's schoolgirl problem (1850)

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- M. Giulietti, A. Sabatini, M. Timpanella, *PIR Codes from Combinatorial Structures, LNCS, 2023*

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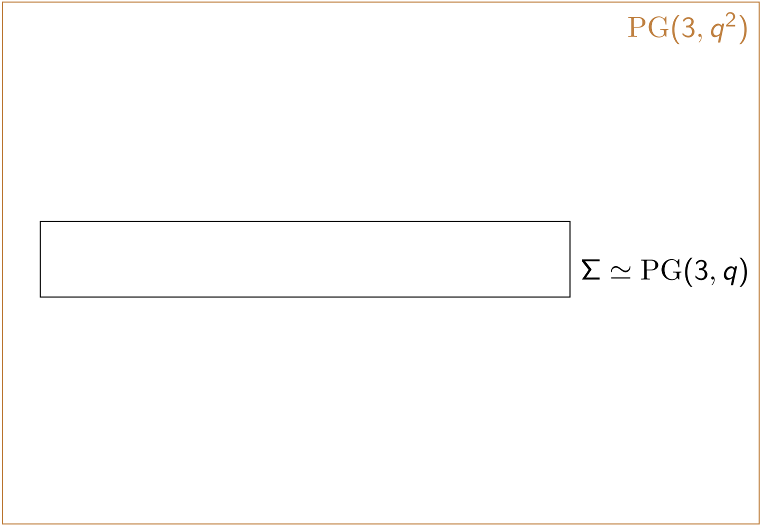

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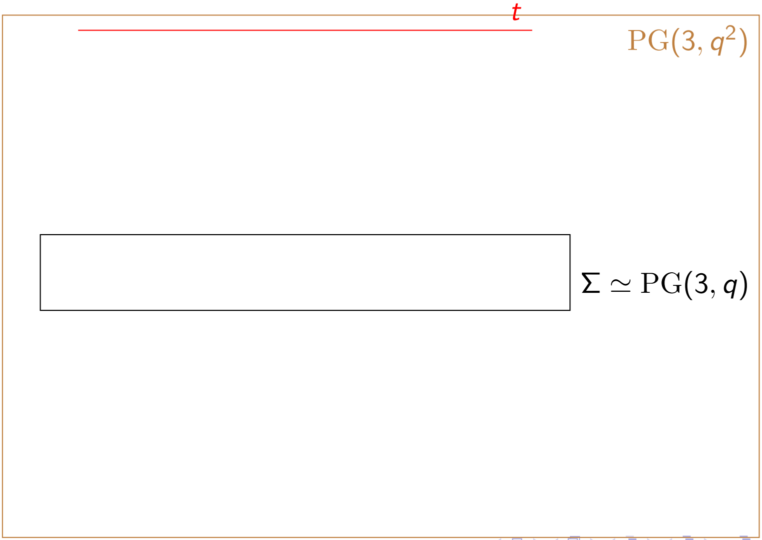

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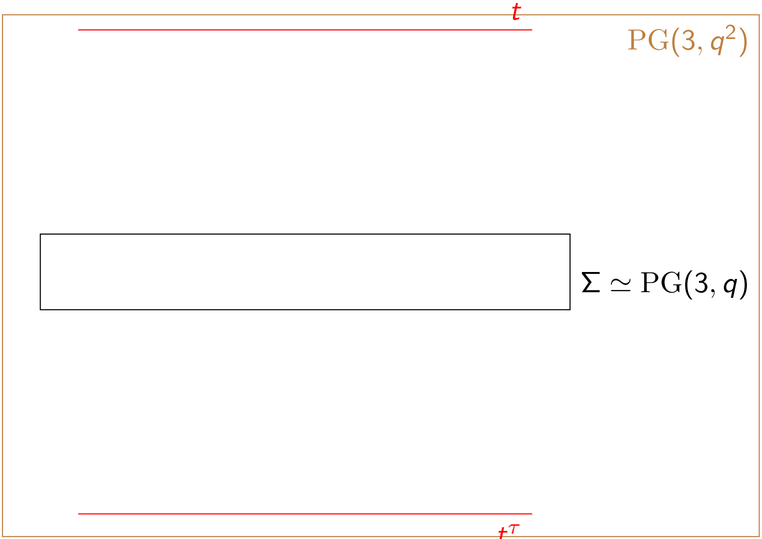

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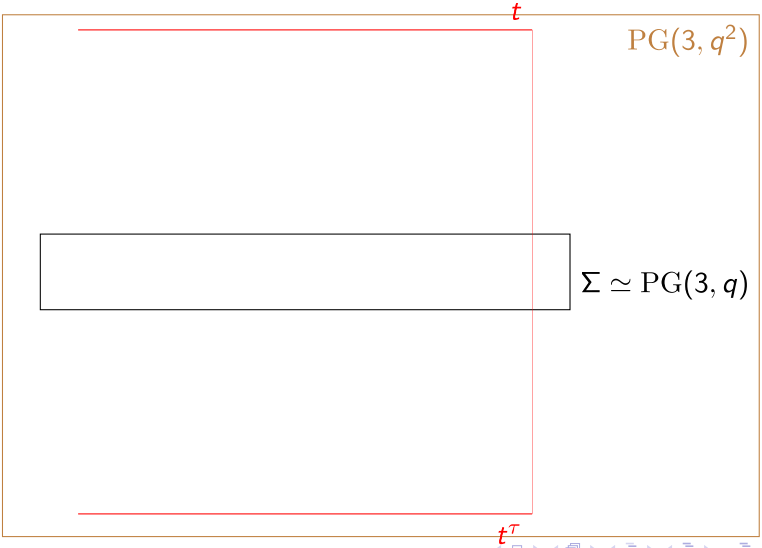

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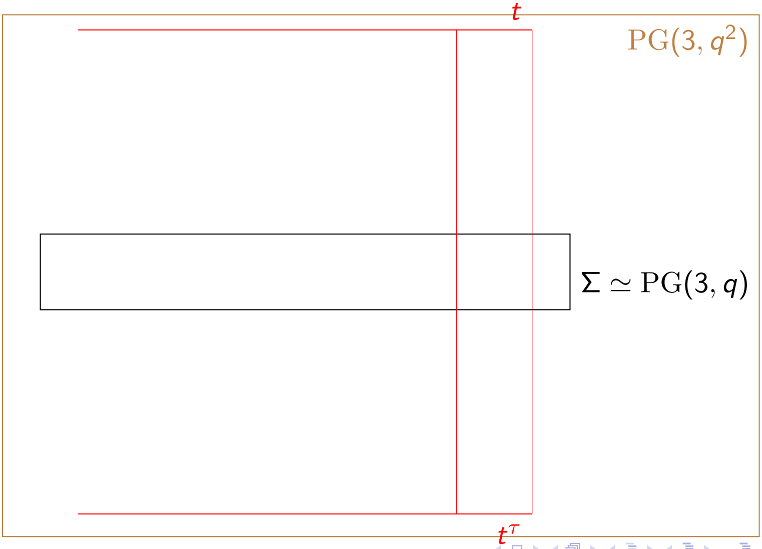

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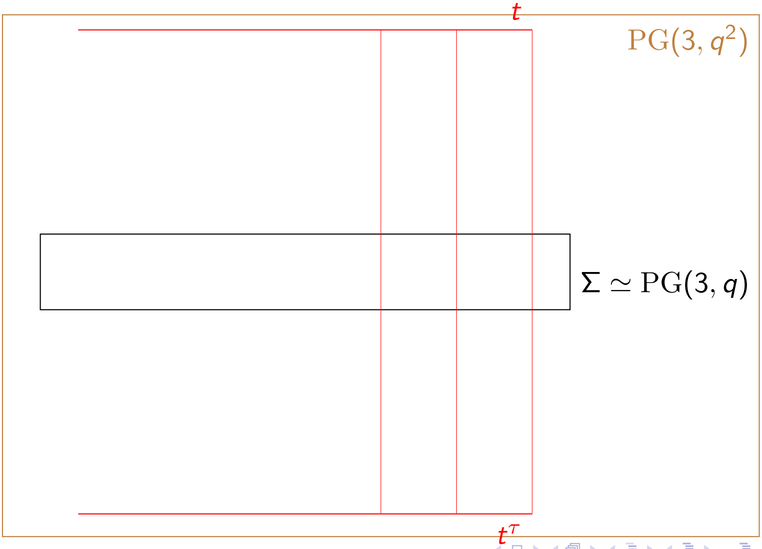

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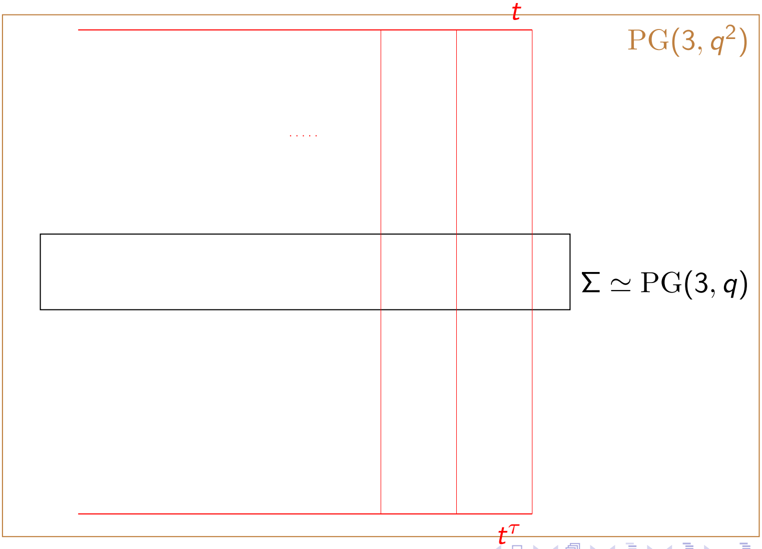

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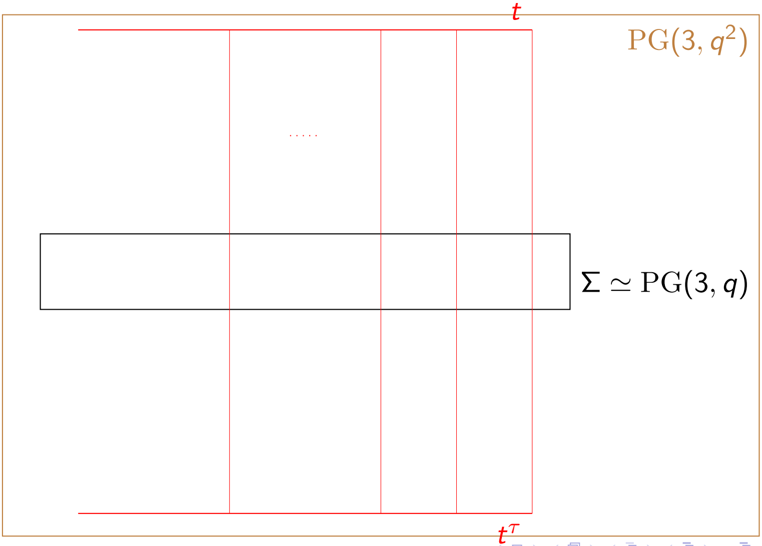

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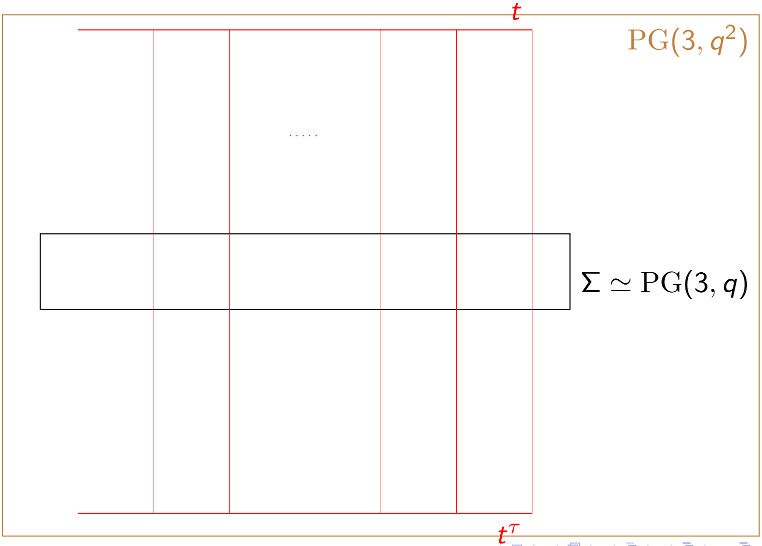

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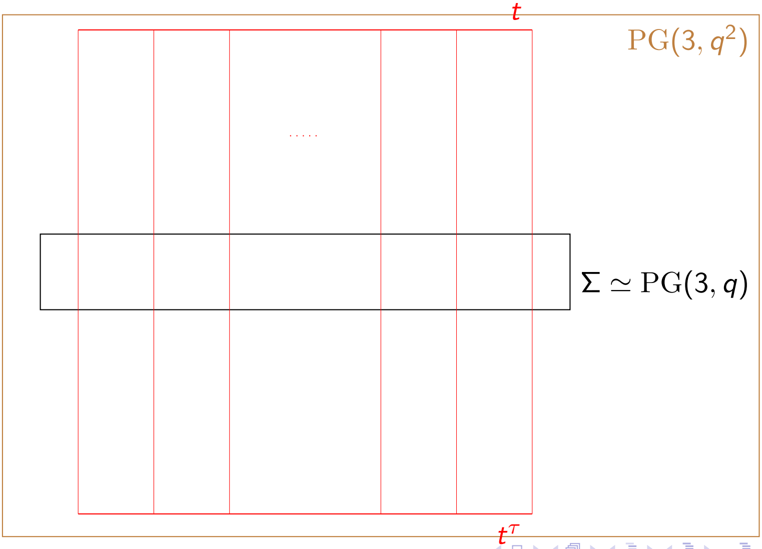

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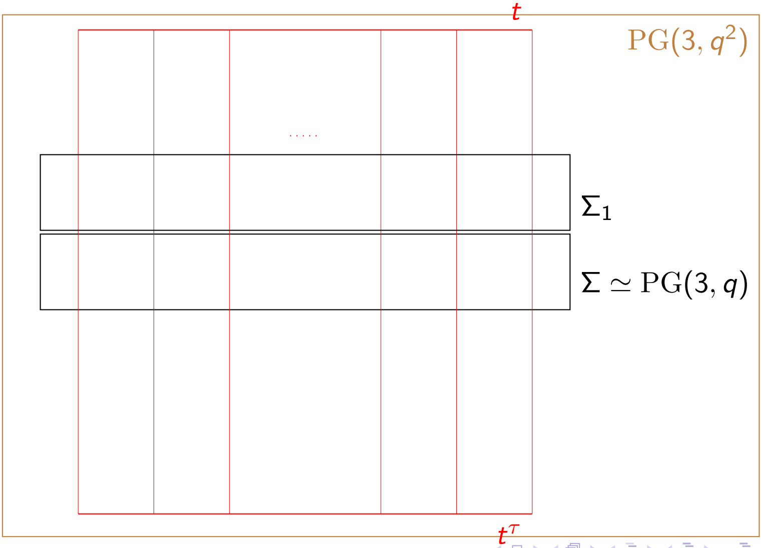

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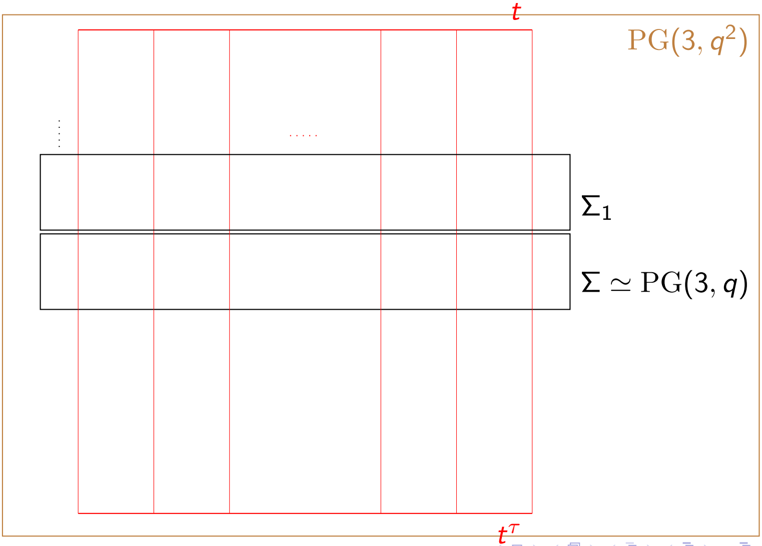

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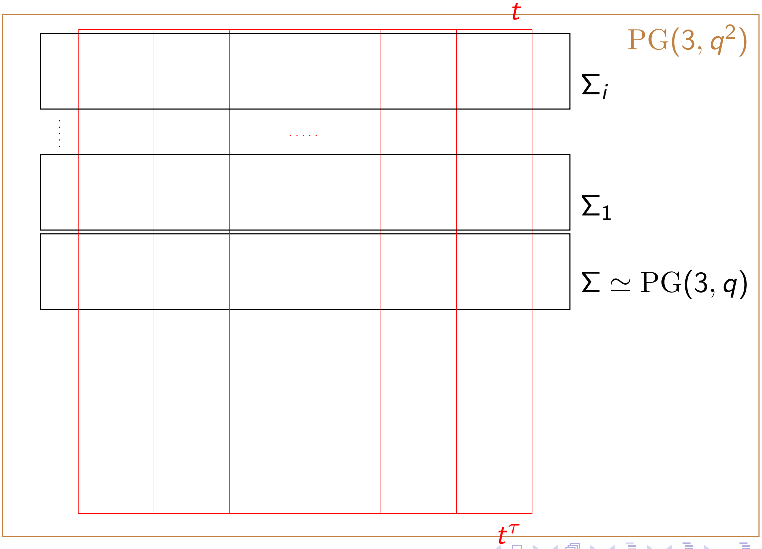

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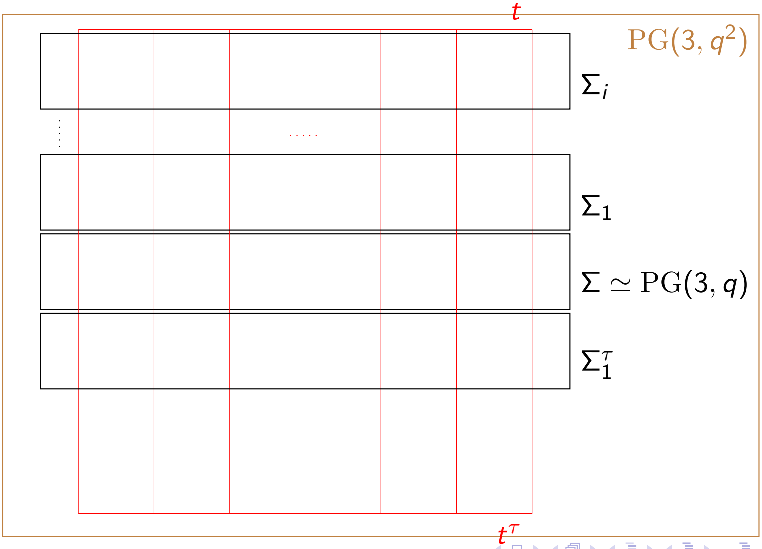

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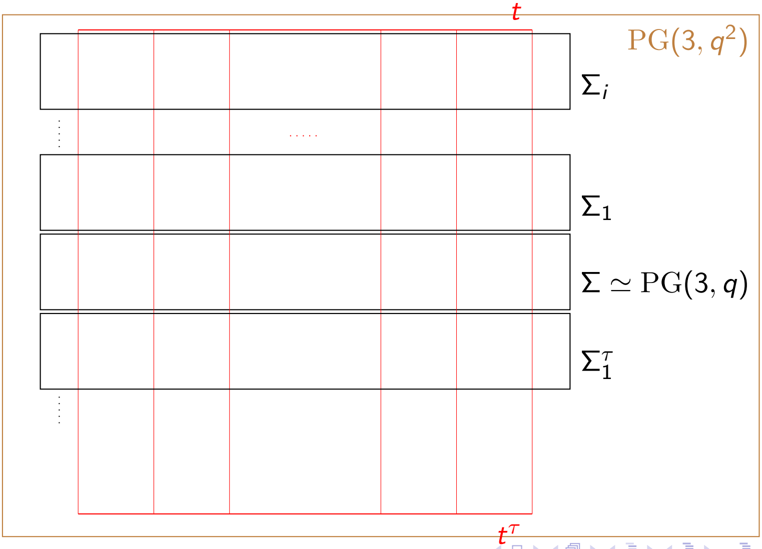

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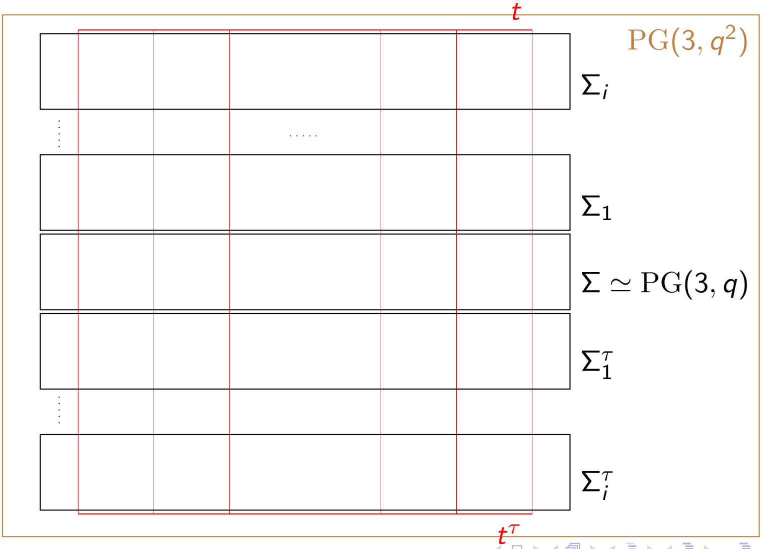
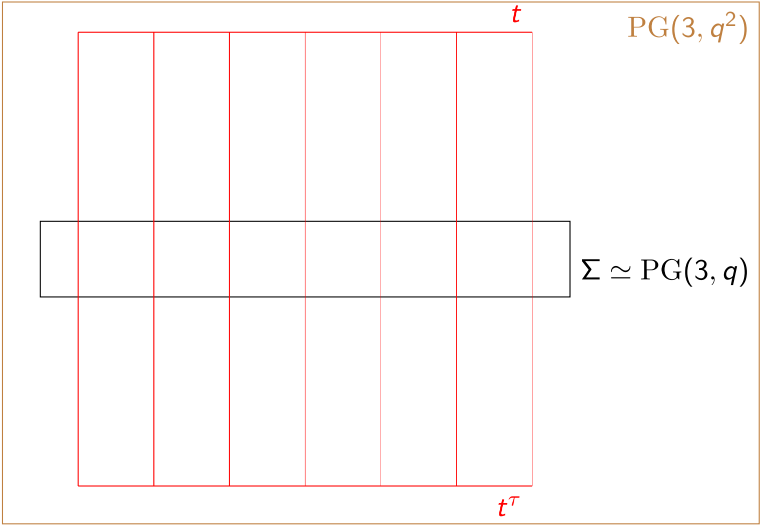
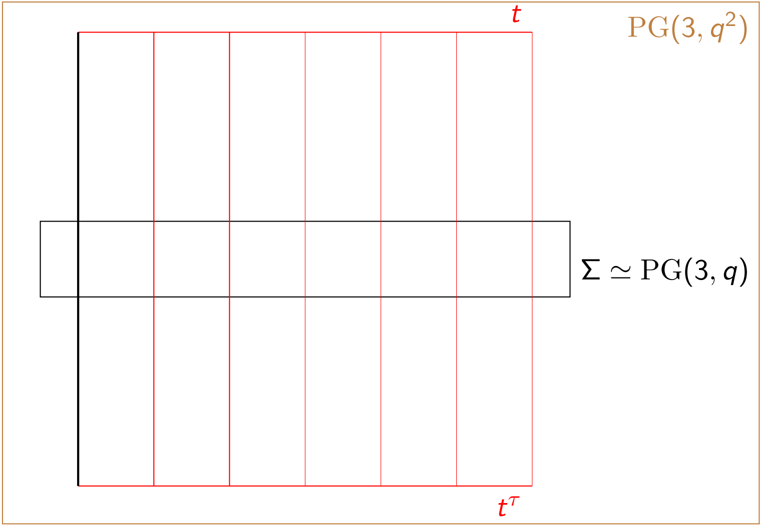



t
$\mathrm{PG}(3, q^2)$
$\Sigma \simeq \mathrm{PG}(3, q)$
t^τ



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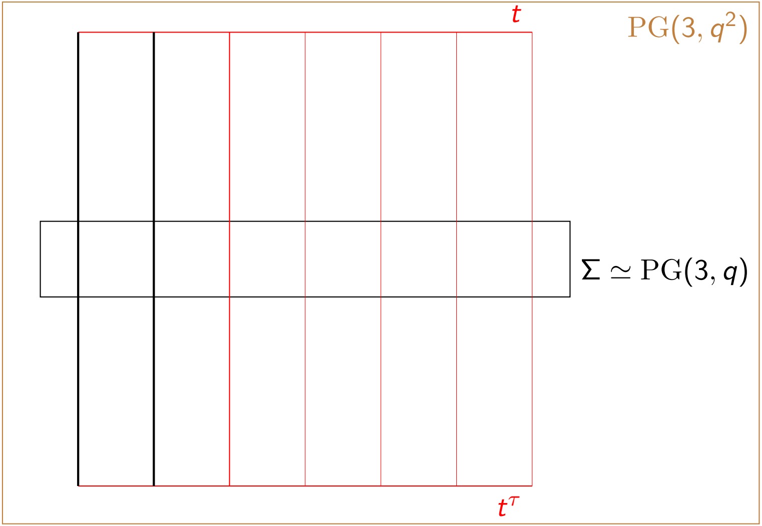
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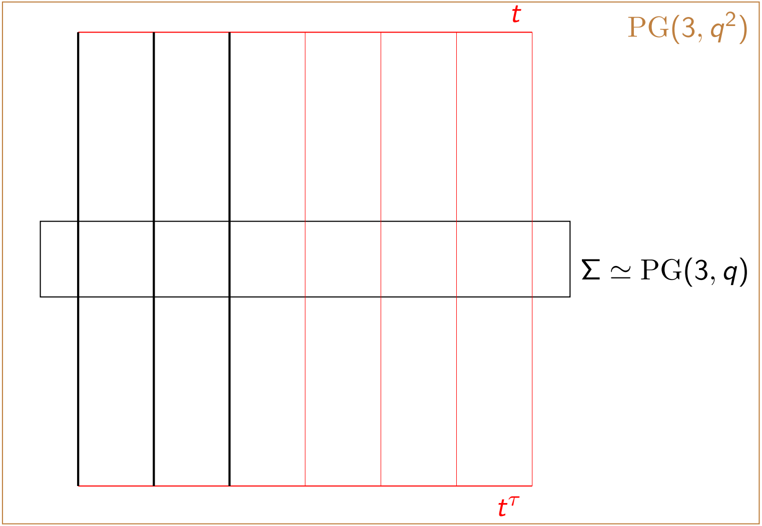

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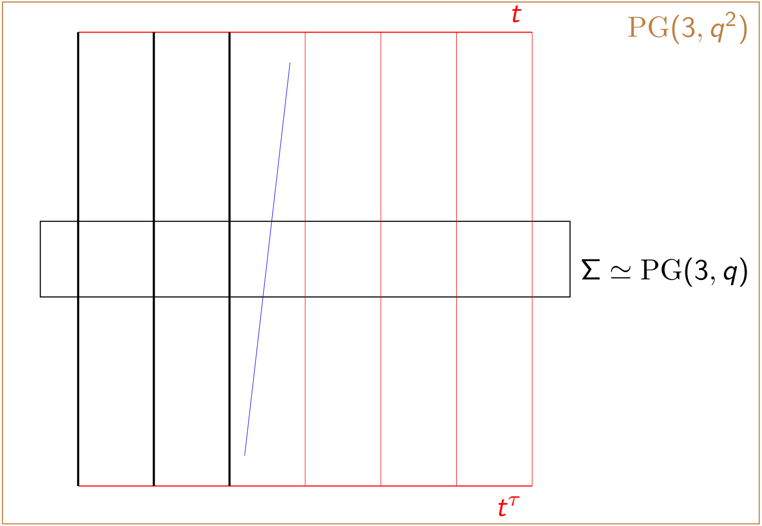

$\mathrm{PG}(3, q^2)$
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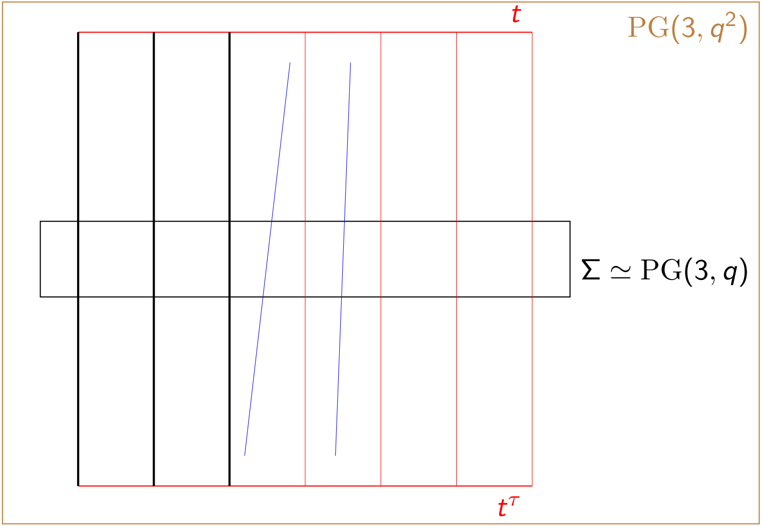
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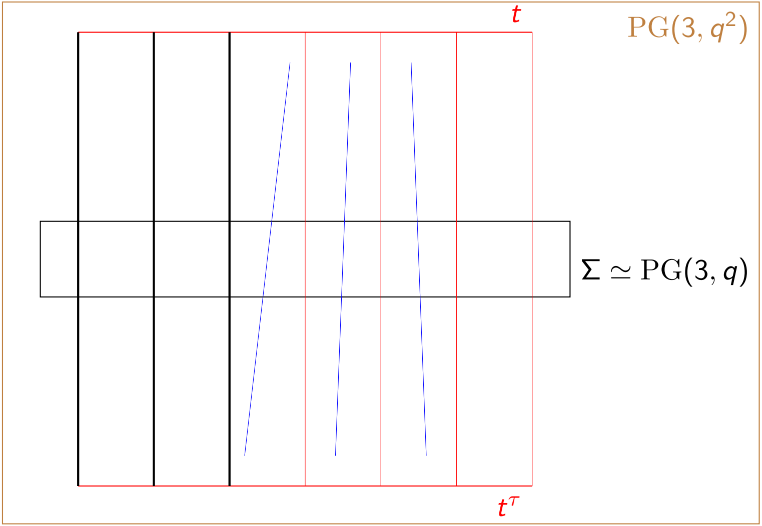
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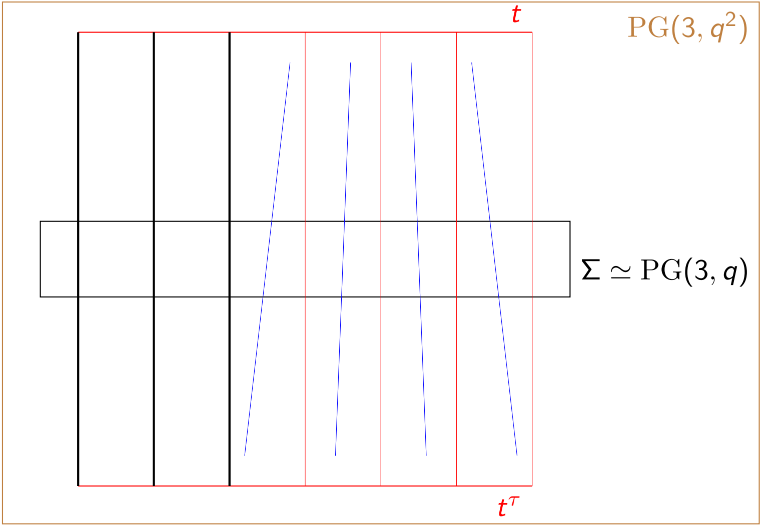
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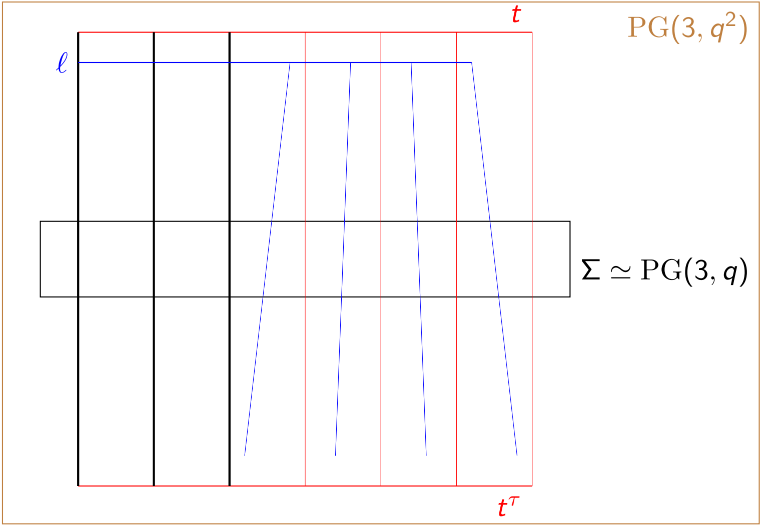
$\mathcal{S}_\ell$ Desarguesian spread of Σ, ℓ, ℓ^τ transversals of $\mathcal{S}_\ell$



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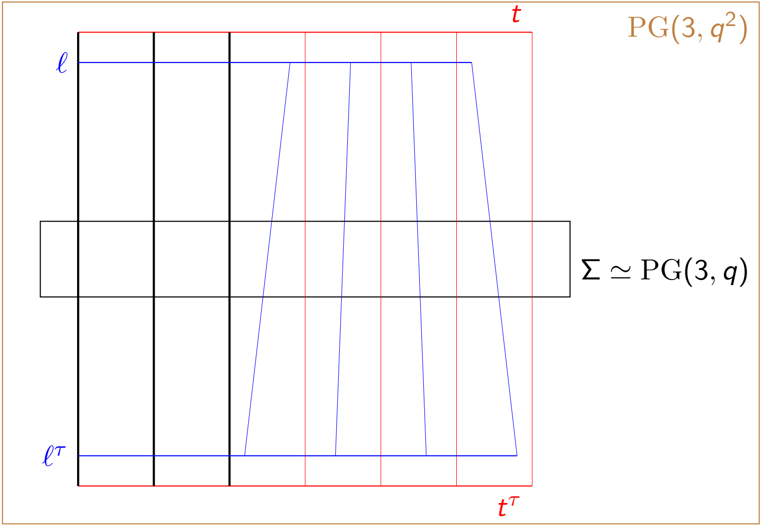
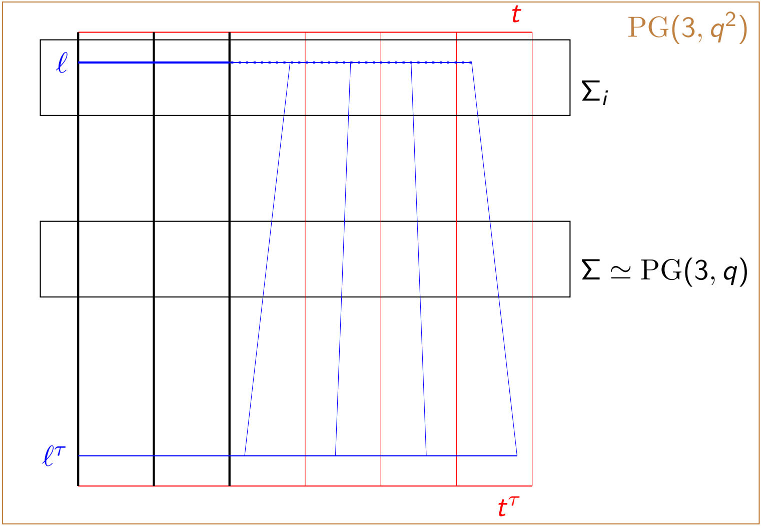


$\mathcal{S}_\ell$ Desarguesian spread of Σ, ℓ, ℓ^τ transversals of $\mathcal{S}_\ell$

$\mathcal{D} \cap \mathcal{S}_\ell$ is a regulus $\mathcal{R}_\ell \iff \ell \cap \Sigma_i$ is a subline.



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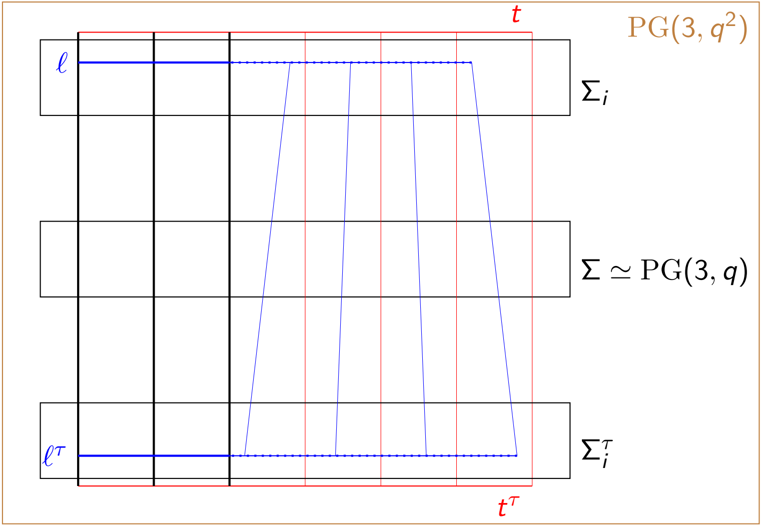
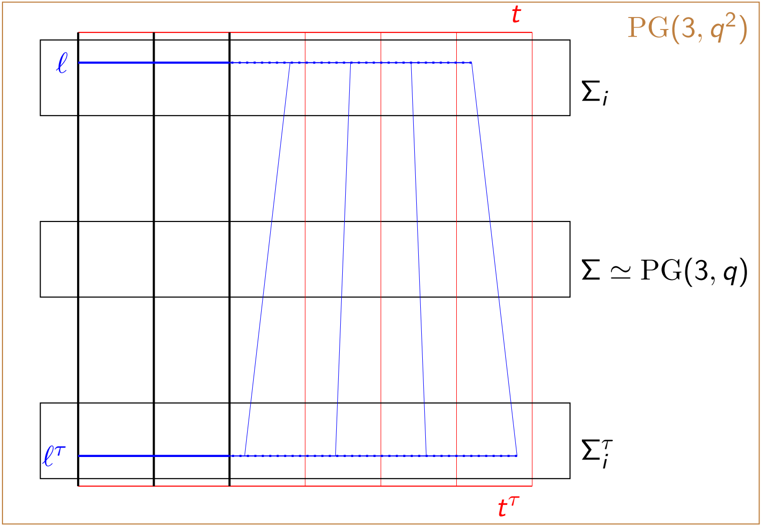
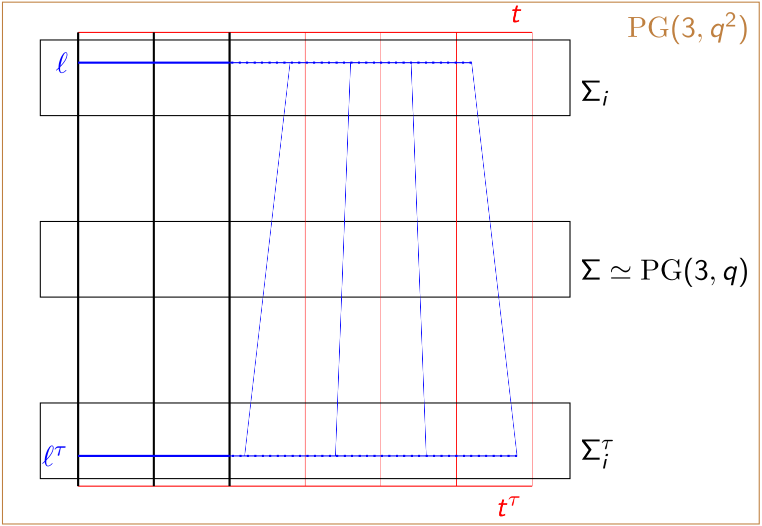


$(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^\circ$ *Hall spread of* Σ

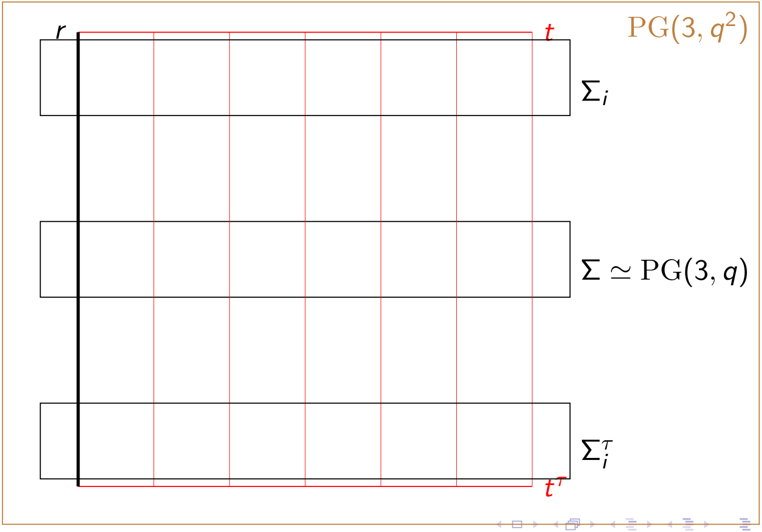


$(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^o$ *Hall spread of* Σ

$\mathcal{D}, (\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^o$ *are pairwise disjoint spreads*

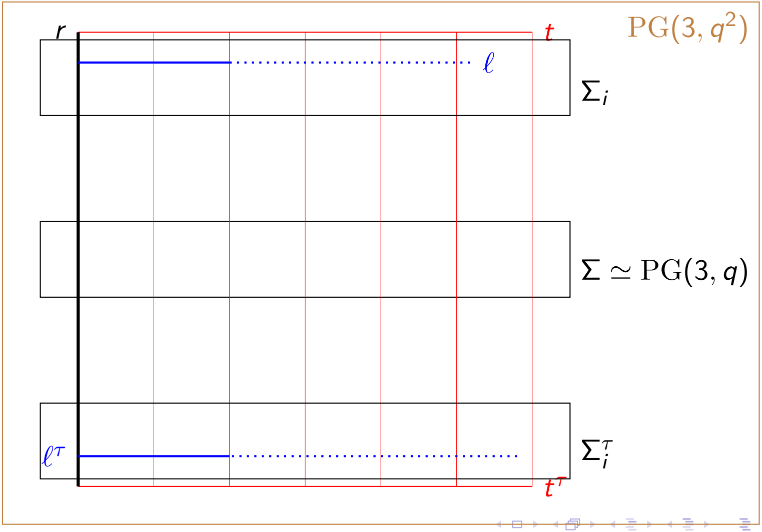


$r \in \mathcal{D}, E \leq Stab_r(Aut(\mathcal{D})),\ E\ elementary\ abelian,\ |E| = q^2$



$r \in \mathcal{D}, E \leq Stab_r(Aut(\mathcal{D}))$, *E elementary abelian*, $|E| = q^2$

ℓ *line of* $\mathrm{PG}(3, q^2)$, $\ell \cap \Sigma = \emptyset$, $\ell \cap \Sigma_i$ *is a subline*



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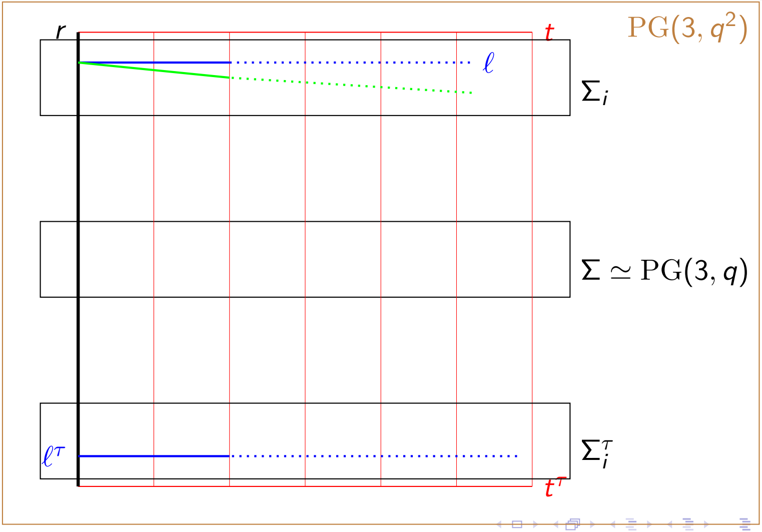
$\ell^E \cup \{r\}$ is a Baer pencil $p(P_{\alpha u}, \pi_{\alpha v})$



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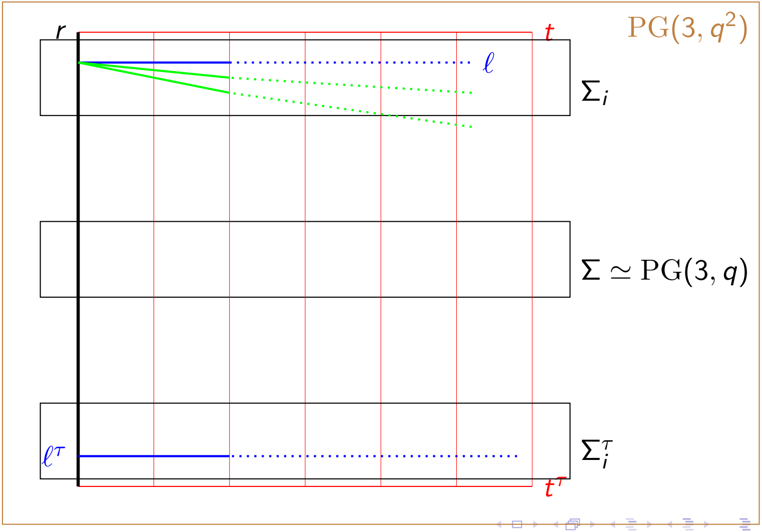

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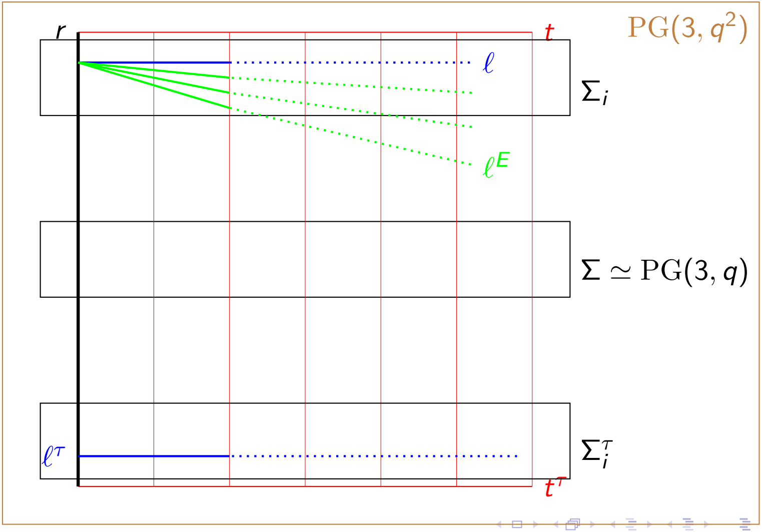

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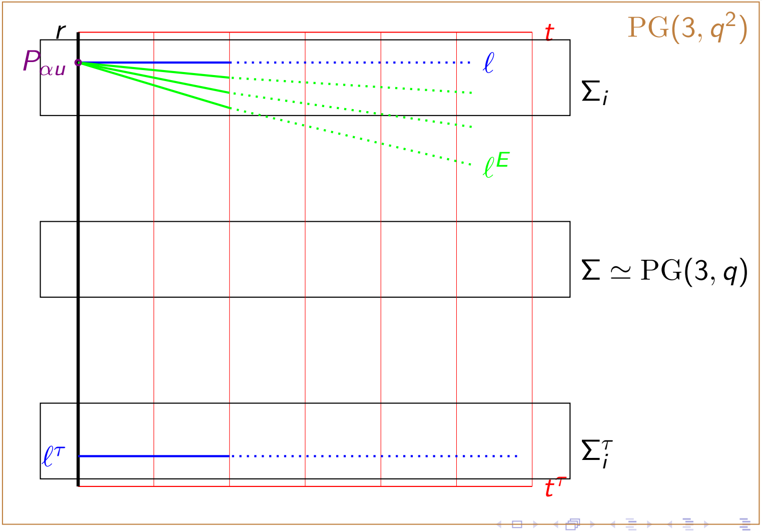

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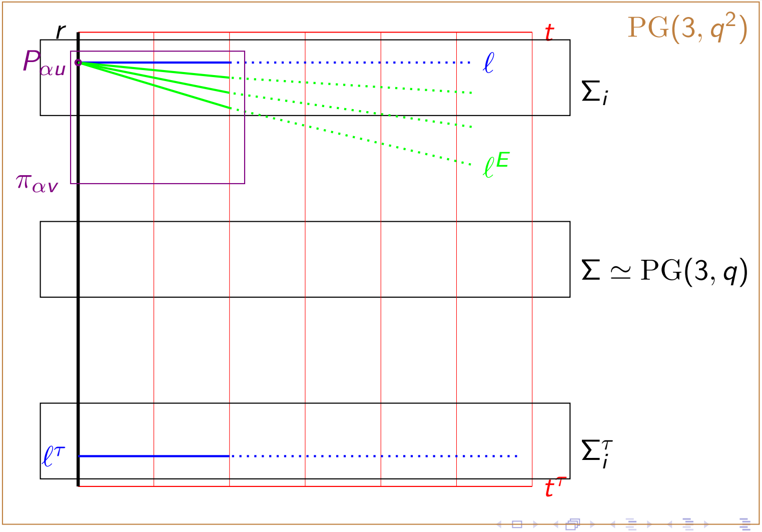
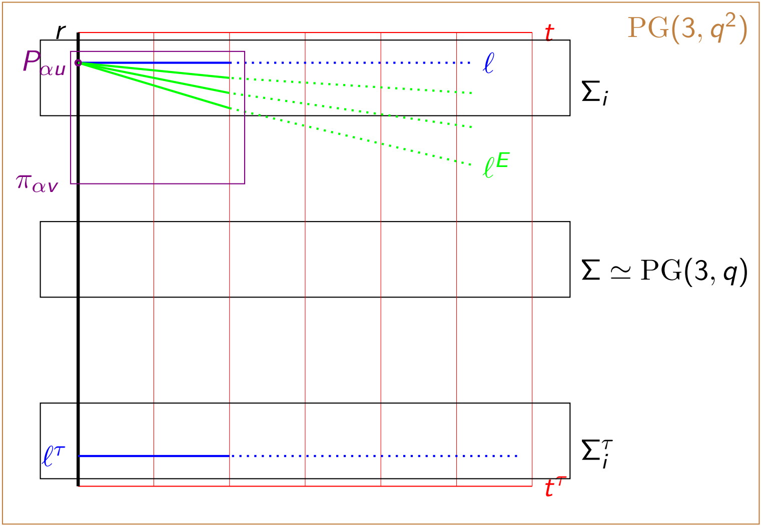
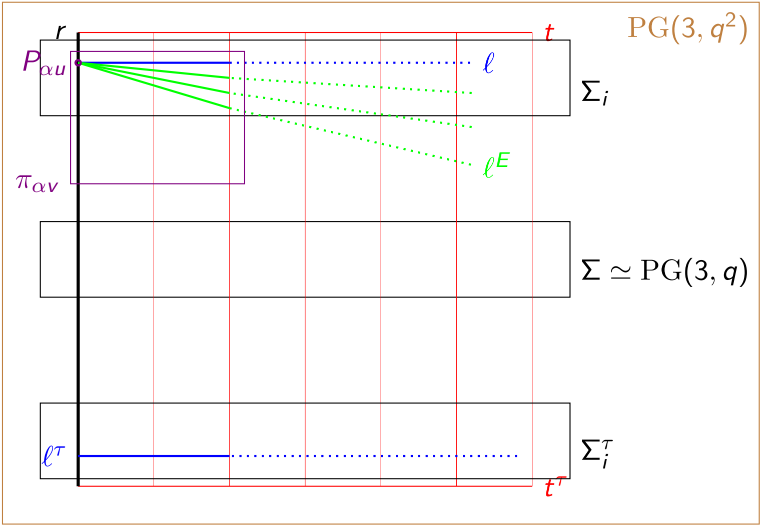


$\{(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^o : \ell \in p(P_{\alpha u,\alpha v}) \setminus \{r\}\}$ *set of q Hall spreads of* Σ

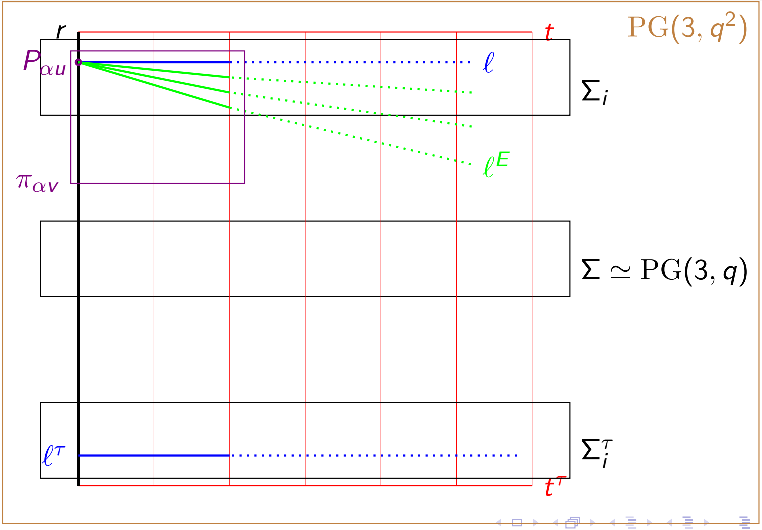


$\{(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}^o_\ell : \ell \in p(P_{\alpha u,\alpha v}) \setminus \{r\}\}$ *set of q Hall spreads of* Σ

$\{\mathcal{D}\} \cup \{(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}^o_\ell : \ell \in p(P_{\alpha u,\alpha v}) \setminus \{r\}\}$ *are pairwise disjoint*



$\Lambda \subset \mathbb{F}_{q^2}$, $|\Lambda| = q-1$, $\{\alpha^{q+1} \colon \alpha \in \Lambda\} = \mathbb{F}_q \setminus \{0\}$



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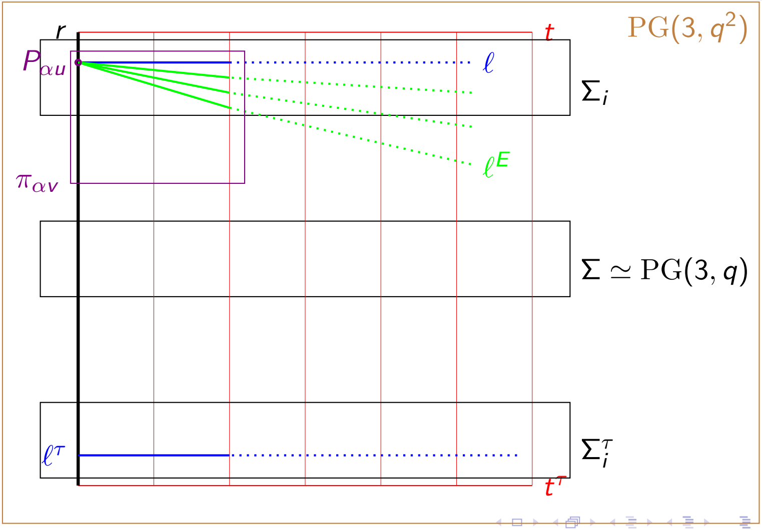
$$\mathcal{I} \subset \Lambda,\ |\mathcal{I}| = \left\lfloor \frac{q-2}{2} \right\rfloor,\ \mathcal{U} = \{u \in \mathbb{F}_{q^2} \mid u^{q+1} = 1\}$$



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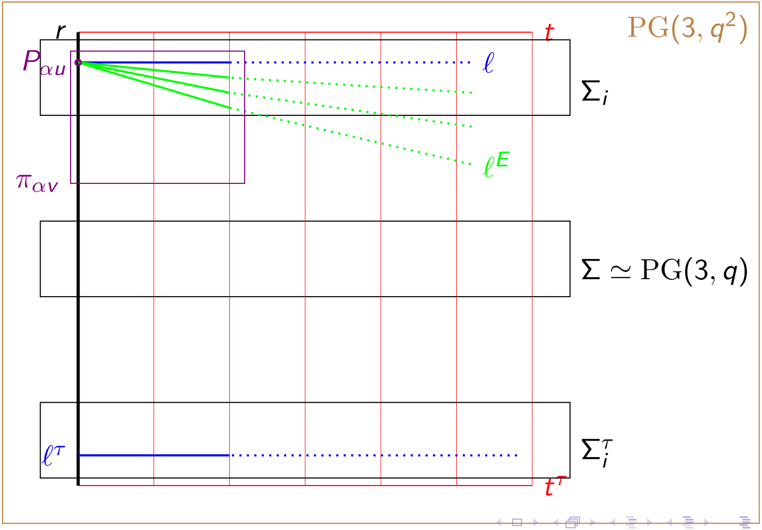
$$P_{\alpha u} = (1, 0, \alpha u, 0),\ \pi_{\alpha v} : X_4 = \alpha v X_2,\ \alpha \in \mathcal{I}, u, v \in \mathcal{U}$$



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$$p\,(P_{\alpha u}, \pi_{\alpha v}) \setminus \{r\}$$

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$$\mathcal{P} = \left\{\left(P_{\alpha_1 u_1}, \pi_{\alpha_1 v_1}\right), \ldots, \left(P_{\alpha_{q+1} u_{q+1}}, \pi_{\alpha_{q+1} v_{q+1}}\right)\right\}$$

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$$\mathcal{L}_{\mathcal{P}} = \left\{p\left(P_{\alpha_i u_i}, \pi_{\alpha_i v_i}\right) \setminus \{r\} \colon \left(P_{\alpha_i u_i}, \pi_{\alpha_i v_i}\right) \in \mathcal{P}\right\}$$

$$P_{\alpha u} = (1, 0, \alpha u, 0),\ \alpha \in \mathcal{I}, u \in \mathcal{U}$$

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$$\Pi_{\mathcal{P}} = \{(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^o \mid \ell \in \mathcal{L}_{\mathcal{P}}\} \cup \{\mathcal{D}\}$$

$$P_{\alpha u} = (1, 0, \alpha u, 0),\ \alpha \in \mathcal{I}, u \in \mathcal{U}$$

$$\pi_{\alpha v} : X_4 = \alpha v X_2,\ \alpha \in \mathcal{I}, v \in \mathcal{U}$$

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$$\Pi_{\mathcal{P}} = \{(\mathcal{S}_\ell \setminus \mathcal{R}_\ell) \cup \mathcal{R}_\ell^o \mid \ell \in \mathcal{L}_{\mathcal{P}}\} \cup \{\mathcal{D}\}$$

How to choose $\mathcal{P}$ such that $\Pi_{\mathcal{P}}$ is a parallelism?

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$
$$(\alpha, u_i, v_i) \neq (\beta, u_j, v_j)$$

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$

$$(\alpha, u_i, v_i) \neq (\beta, u_j, v_j)$$

Property ①

$$\mathcal{R}_{\ell_i} \neq \mathcal{R}_{\ell_j} \iff u_i v_j - u_j v_i \neq 0.$$

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$
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Property ②

$$\left(\mathcal{S}_{\ell_i} \setminus \mathcal{R}_{\ell_i}\right) \cap \left(\mathcal{S}_{\ell_j} \setminus \mathcal{R}_{\ell_j}\right) = \emptyset \iff \alpha u_i \left(\beta v_j\right)^q - \left(\alpha v_i\right)^q \beta u_j \neq 0.$$

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$
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Good set

$$\mathcal{P} = \left\{\left(P_{\alpha_1 u_1}, \pi_{\alpha_1 v_1}\right), \ldots, \left(P_{\alpha_{q+1} u_{q+1}}, \pi_{\alpha_{q+1} v_{q+1}}\right)\right\}$$

$$\ell_i \in p\left(P_{\alpha u_i}, \pi_{\alpha v_i}\right),\ \ell_j \in p\left(P_{\beta u_j}, \pi_{\beta v_j}\right)$$
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Property ①

$$\mathcal{R}_{\ell_i} \neq \mathcal{R}_{\ell_j} \iff u_i v_j - u_j v_i \neq 0.$$

Property ②

$$\left(\mathcal{S}_{\ell_i} \setminus \mathcal{R}_{\ell_i}\right) \cap \left(\mathcal{S}_{\ell_j} \setminus \mathcal{R}_{\ell_j}\right) = \emptyset \iff \alpha u_i \left(\beta v_j\right)^q - \left(\alpha v_i\right)^q \beta u_j \neq 0.$$

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 # good sets:

$$\begin{cases} \left(\frac{q-2}{2}\right)^{q+1} (q+1)! & \textit{if } q \textit{ is even}, \\ \left(\frac{(q+1)(q-3)}{16}\right)^{\frac{q+1}{2}} \prod_{i=0}^{\frac{q-1}{2}} (q+1-2i)^2 & \textit{if } q \equiv 1 \pmod 4, \\ \left(\frac{q-1}{4}\right)^{q+1} \prod_{i=0}^{\frac{q-1}{2}} (q+1-2i)^2 & \textit{if } q \equiv 3 \pmod 4. \end{cases}$$

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THANK YOU