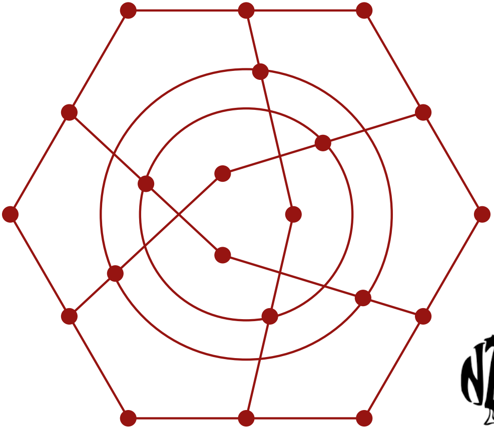
Characterising the natural embedding of the twisted triality hexagons



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September 2025, Irsee

Joint work with Geertrui Van de Voorde

Generalised hexagons

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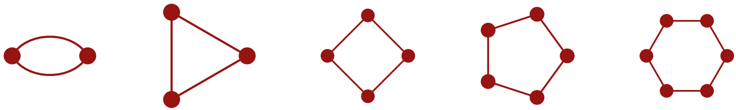

Definition.

Generalised hexagon, Γ:

Point-line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ s. t.:

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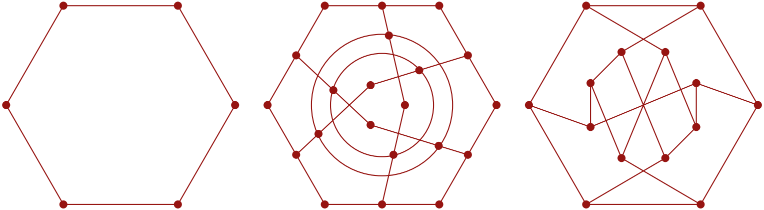
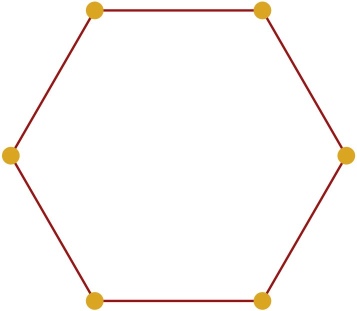
Definition.
Order (s, t):

- line $\implies s + 1$ points,
- point $\implies t + 1$ lines.

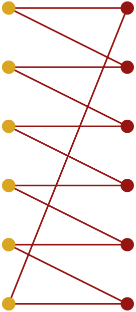
Generalised hexagons



Order $(1, 1)$ Order $(2, 1)$ Order $(1, 2)$



Point-line geometry



Incidence graph

Alternate definition.

A generalised hexagon Γ is a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ such that the incidence graph of Γ is connected and has:

- diameter 6,
- girth 12,
- every vertex has degree at least 2.

Definition.
Thick generalised hexagon: Generalised hexagon of order (s, t) with $s > 1$ and $t > 1$.

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Known thick generalised hexagons:

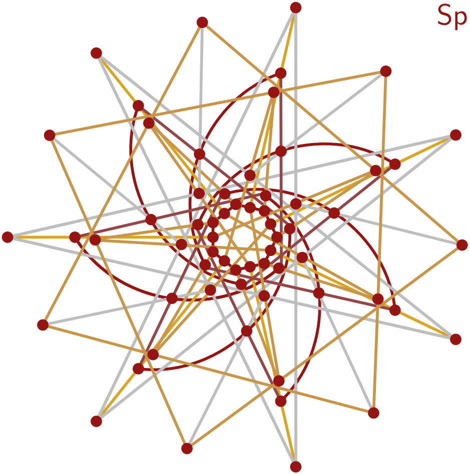
- split Cayley hexagons, order (q, q),
- twisted triality hexagons, order (q^3, q),

and their *duals*.

Split Cayley hexagons

Split Cayley hexagons

Split Cayley hexagon



H(2)

$\mathsf{H} :=$ split Cayley hexagon $\mathsf{H}(q) = (\mathcal{P}^{\mathsf{H}}, \mathcal{L}^{\mathsf{H}}, \mathtt{I})$ in $\mathsf{PG}(6, q)$.

- Points: all the points of $\mathsf{Q}(6, q)$.
- Lines: subset of the lines of $\mathsf{Q}(6, q)$.

H := split Cayley hexagon $\mathsf{H}(q) = (\mathcal{P}^{\mathsf{H}}, \mathcal{L}^{\mathsf{H}}, \mathtt{I})$ in $\mathsf{PG}(6, q)$.

- Points: all the points of $\mathsf{Q}(6, q)$.
- Lines: subset of the lines of $\mathsf{Q}(6, q)$.

Let x be a point of H.

- (Flat) The set of points collinear with x in H is contained in a plane of $\mathsf{PG}(6, q)$.
- (Weak) The set of points not opposite x in H is contained in a hyperplane of $\mathsf{PG}(6, q)$.

- (Pt) Point: 0 or $q+1$ incident elements of $\mathcal{L}^{\mathsf{H}}$
- (Pl) Plane: $0, 1$ or $q+1$ elements of $\mathcal{L}^{\mathsf{H}}$.
- (Sd) Solid: $0, 1, q+1$ or $2q+1$ elements of $\mathcal{L}^{\mathsf{H}}$.
- (4d) 4-spaces: at most $q^3 - q^2 + 4q$ elements of $\mathcal{L}^{\mathsf{H}}$.
- (To) $|\mathcal{L}^{\mathsf{H}}| \leq q^5 + q^4 + q^3 + q^2 + q + 1$.

Theorem (Ihringer (2014)).
If $\mathcal{L}$ is a set of lines of $\mathrm{PG}(6,q)$ then $\mathcal{L}$ satisfies (Pt), (Pl), (Sd), (4d) and (To), if and only if it is the line set of a naturally embedded split Cayley hexagon $\mathrm{H}(q)$ in $\mathrm{PG}(6,q)$.

F. Ihringer, *A characterization of the natural embedding of the split Cayley hexagon in PG(6,q) by intersection numbers in finite projective spaces of arbitrary dimension*, Discrete Mathematics, v. 314, p. = 42-49, 2014, ISSN 0012-365X.

J. A. Thas, H. Van Maldeghem, *A characterization of the natural embedding of the split Cayley hexagon H(q) in PG(6,q) by intersection numbers*, European Journal of Combinatorics, v. 29, i. 6, 2008, p. 1502-1506, ISSN 0195-6698.

Twisted triality hexagons

$\mathsf{T} :=$ twisted triality hexagon $\mathsf{T}(q^3, q) = (\mathcal{P}^{\mathsf{T}}, \mathcal{L}^{\mathsf{T}}, \mathtt{I})$ in $\mathrm{PG}(7, q^3)$.

- Points: subset of the points of $\mathrm{Q}^+(7, q^3)$.
- Lines: subset of the lines of $\mathrm{Q}^+(7, q^3)$.

Twisted triality hexagons

T := twisted triality hexagon $\mathsf{T}(q^3, q) = (\mathcal{P}^{\mathsf{T}}, \mathcal{L}^{\mathsf{T}}, \mathtt{I})$ in PG$(7, q^3)$.

- Points: subset of the points of $\mathsf{Q}^+(7, q^3)$.
- Lines: subset of the lines of $\mathsf{Q}^+(7, q^3)$.

Let x be a point of T.

- (Flat) The set of points collinear with x in T is contained in a plane of PG$(7, q^3)$.
- (Weak) The set of points not opposite x in T is contained in a hyperplane of PG$(7, q^3)$.

Definition.
Let U be a subspace of $\mathrm{PG}(7, q^3)$.
Let x be a point of T in U.

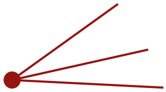
- x is $\mathcal{L}^{\mathrm{T}}$-**isolated** in U if no line of $\mathcal{L}^{\mathrm{T}}$ through x is in U.
- x is $\mathcal{L}^{\mathrm{T}}$-**ideal** in U if all lines of $\mathcal{L}^{\mathrm{T}}$ through x are in U.

Being isolated/ideal depends on the subspace in which we consider the point!

Definition.
An n-dimensional subspace U of $\mathrm{PG}(7, q^3)$ is $\mathcal{L}^{\mathsf{T}}$-**supported** if all lines of $\mathcal{L}^{\mathsf{T}}$ in U span the space U.

(Pt) Point: 0 or $q+1$ incident elements of $\mathcal{L}^{\mathsf{T}}$.
(To) $|\mathcal{L}^{\mathsf{T}}| \leq q^9 + q^8 + q^5 + q^4 + q + 1$.

Lemma.
Every $\mathcal{L}^{\mathsf{T}}$-supported plane of $\mathrm{PG}(7, q^3)$ is incident with $q+1$ lines of $\mathcal{L}^{\mathsf{T}}$.



$q+1$ lines
0 isolated, 1 ideal

$\Longrightarrow$ **(Pl)** Plane: $0, 1$ or $q+1$ incident elements of $\mathcal{L}^{\mathsf{T}}$.

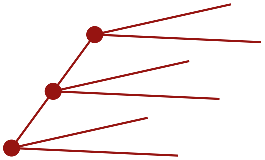
Lemma.
Let Σ be an $\mathcal{L}^{\mathsf{T}}$-supported solid of $\mathrm{PG}(7, q^3)$. Then, Σ contains either $q+1$ or $2q+1$ elements of $\mathcal{L}^{\mathsf{T}}$.

$q+1$ lines
0 isolated, 0 ideal

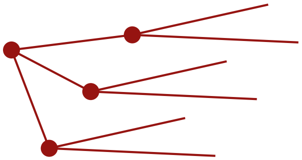
$2q+1$ lines
0 isolated, 2 ideal

$\Longrightarrow$ **(Sd)** Solid: $0, 1, q+1$ or $2q+1$ incident elements of $\mathcal{L}^{\mathsf{T}}$.

Lemma.
Let U be an $\mathcal{L}^{\mathsf{T}}$-supported 4-dimensional subspace of $\mathrm{PG}(7, q^3)$. Then, U contains either q^2+q+1 or q^2+2q+1 elements of $\mathcal{L}^{\mathsf{T}}$.



q^2+q+1 lines
0 isolated, $q+1$ ideal

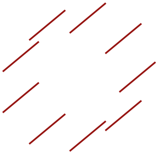


q^2+2q+1 lines
0 isolated, $q+2$ ideal

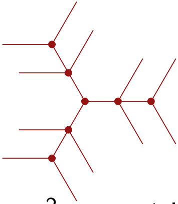
$\Longrightarrow$ **(4d)** 4-dim subspace: at most q^2+2q+1.

Lemma.
Let V be an $\mathcal{L}^{\mathsf{T}}$-supported 5-dimensional subspace of $\mathrm{PG}(7, q^3)$.
Then, V contains either
$q^3 + 1, q^3 + q^2 + q + 1, q^3 + 2q^2 + 2q + 1$ or $q^4 + q + 1$
elements of $\mathcal{L}^{\mathsf{T}}$.

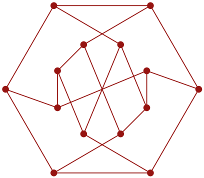
Twisted triality hexagons



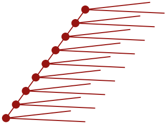
$q^3 + 1$ lines
0 isolated, 0 ideal



$q^3 + q^2 + q + 1$ lines
0 isolated, $q^2 + q + 1$ ideal

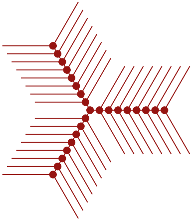


$q^3 + 2q^2 + 2q + 1$ lines
0 isolated, $2q^2 + 2q + 2$ ideal

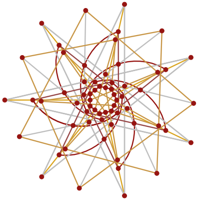


$q^4 + q + 1$ lines
0 isolated, $q^3 + 1$ ideal

Lemma.
Let W be an $\mathcal{L}^\mathsf{T}$-supported hyperplane of $\mathrm{PG}(7, q^3)$. Then, W contains either $q^5 + q^4 + q + 1$ or $q^5 + q^4 + q^3 + q^2 + q + 1$ elements of $\mathcal{L}^\mathsf{T}$.



$q^5 + q^4 + q + 1$ lines



$q^5 + q^4 + q^3 + q^2 + q + 1$ lines

A set of lines in PG$(7, q^3)$

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Set of lines $\mathcal{L}$ of PG$(7, q^3)$ such that:

- (Pt) Point: 0 or $q+1$ incident elements of $\mathcal{L}$
- (Pl) Plane: $0, 1$ or $q+1$ elements of $\mathcal{L}$.
- (Sd) Solid: $0, 1, q+1$ or $2q+1$ elements of $\mathcal{L}$.
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- (To) $|\mathcal{L}^{\mathrm{H}}| \leq q^5+q^4+q^3+q^2+q+1$.

A set of lines in PG$(7, q^3)$

Combinatorially from (Pt), (Pl), (Sd) and (4d):

- No three lines of $\mathcal{L}$ form a triangle.
- No four lines of $\mathcal{L}$ form a quadrangle.
- No five lines of $\mathcal{L}$ form a pentagon.

Together with property (To):

Lemma.
The set $\mathcal{L}$ determines a generalised hexagon of order (q^3, q).

A set of lines in $\mathrm{PG}(7, q^3)$

Theorem (Thas & Van Maldeghem (1998)).
If a thick generalized hexagon Γ of order (s, t) is flatly and fully embedded in $\mathrm{PG}(d, s)$, then $d \in \{4, 5, 6, 7\}$ and $t \leq s$. Also, if $d = 7$, then $\Gamma \cong \mathrm{T}(s, \sqrt[3]{s})$ and the embedding is natural. If $d = 6$ and $t^5 > s^3$, then $\Gamma \cong \mathrm{H}(s)$ and the embedding is natural. If $d = 5$ and $s = t$, then $\Gamma \cong \mathrm{H}(s)$ with s even and the embedding is natural.

J. A Thas, H. Van Maldeghem, *Flat Lax and Weak Lax Embeddings of Finite Generalized Hexagons*, European Journal of Combinatorics, v. 19, i. 6, 1998, p. 733-751, ISSN 0195-6698.

Conclusion: our contribution

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Theorem (P. & Van de Voorde (2025)).
The line set $\mathcal{L}$ of a regularly embedded twisted triality hexagon $\mathsf{T}(q^3, q)$ in $\mathrm{PG}(7, q^3)$ satisfies the properties (Pt), (Pl), (Sd), (4d) and (To).

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Theorem (P. & Van de Voorde (2025)).
Let $\mathcal{L}$ be a set of lines of $\mathrm{PG}(7, q^3)$. If $\mathcal{L}$ satisfies the properties (Pt), (Pl), (Sd), (4d) and (To), then $\mathcal{L}$ is the line set of a regularly embedded twisted triality hexagon $\mathsf{T}(q^3, q)$ in $\mathrm{PG}(7, q^3)$.

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Future work: weaken/change the hypothesis to draw the same conclusion.

Thank you for your attention!