Additive Codes and Finite Geometries

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Definition

A *linear code* of length n, dimension k and distance d over $\mathbb{F}_q$ is a k dimensional linear subspace C of $\mathbb{F}_q^n$ where d is the minimum weight of any non-zero $c \in C$. We denote such a code by $[n, k, d]_q$

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An *additive code* of length n over $\mathbb{F}_{q^h}$ is a subset C of $\mathbb{F}_{q^h}^n$ with the property that for all $u, v \in C$ the sum $u + v \in C$.

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An *additive code* of length n over $\mathbb{F}_{q^h}$ is a subset C of $\mathbb{F}_{q^h}^n$ with the property that for all $u, v \in C$ the sum $u + v \in C$.

- We use the notation $[n, r/h, d]_q^h$ to denote an additive code of length n over $\mathbb{F}_{q^h}$, of size q^r and minimum distance d, which is linear over $\mathbb{F}_q$.
- Note when $h = 1$ we end up with a linear code.

Geometry of Additive Codes

An $[n, r/h, d]_q^h$ additive code is generated by a $r \times n$ matrix G where each column U is in $\mathbb{F}_{q^h}^r$

$$G = \begin{bmatrix} | & | & & | \\ U_1 & U_2 & \dots & U_n \\ | & | & & | \end{bmatrix}.$$

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Let $\{e_1, \dots, e_h\}$ be a basis for $\mathbb{F}_{q^h}$ over $\mathbb{F}_q$, then

$$U_i = \sum_{j=1}^{h} u_j e_j \quad \text{for} \quad u_j \in \mathbb{F}_q^r$$

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Let $\pi_i = \langle u_1 \dots u_h \rangle$ be a subspace of $PG(r-1, q)$ for each U_i and $\mathcal{X}_G = \{\pi_1 \dots \pi_n\}$ be the multiset of subspaces

Relationship with Projective Geometries

- There exists a one-to-one relationship between codewords of the additive code and hyperplanes, such that a coordinate c_i of the codeword is 0 if and only if π_i is in the corresponding hyperplane.

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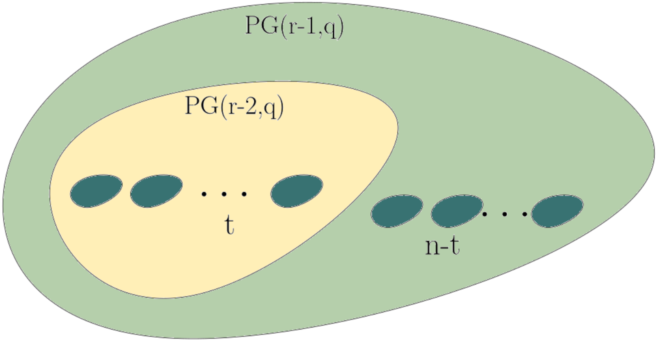
- There exists a one-to-one relationship between codewords of the additive code and hyperplanes, such that a coordinate c_i of the codeword is 0 if and only if π_i is in the corresponding hyperplane.
- If a hyperplane contains t members of $\mathcal{X}_G = \{\pi_1 \ldots \pi_n\}$ it has weight $n - t$.

Relationship with projective Geometries



Projective Systems

Definition

A *projective* $h-(n,r,d)_q$ *system* is a multiset S of n subspaces of $\mathrm{PG}(r-1,q)$ of dimension at most $h-1$ such that each hyperplane of $\mathrm{PG}(r-1,q)$ contains at most $n-d$ elements of S, and some hyperplane contains exactly $n-d$ elements of S.

Theorem

If C is an additive $[n,r/h,d]_q^h$ code, then $\mathcal{X}(\mathcal{C})$ is a projective $h-(n,r,d)_q$ system, and conversely, each projective $h-(n,r,d)_q$ system defines an additive $[n,r/h,d]_q^h$ code.

Mathon Example

Theorem (Mathon 2003)

There exists a set of 21 *lines* $\mathcal{X}$ *in* $PG(5,3)$ *such that every plane contains* 0 *or* 3 *lines of* $\mathcal{X}$.

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Corollary

There exists a $[21, 3, 18]_3^2$ additive code

Griesmer Bound

Theorem (Griesmer 1960)

If there is an $[n,k,d]_q$ *linear code then*

$$n \geqslant \sum_{j=0}^{k-1} \left\lceil \frac{d}{q^j} \right\rceil .$$

Griesmer Bound

Theorem (Griesmer 1960)

If there is an $[n, k, d]_q$ linear code then

$$n \geqslant \sum_{j=0}^{k-1} \left\lceil \frac{d}{q^j} \right\rceil.$$

This bound can be reformulated as follows,

$$n \geqslant k + d - m + \sum_{j=1}^{m-1} \left\lceil \frac{d}{q^j} \right\rceil,$$

where $m \leqslant r - 1$ is such that $q^{k-1} < d \leqslant q^m$.

Additive Griesmer Bound 1.

Theorem (Ball, Lavrauw, P. 2024)

If there is an $[n, r/h, d]_q^h$ *additive code then*

$$n \geqslant \lceil r/h \rceil + d - m - 2 + \lceil \frac{d}{f(q,m)} \rceil,$$

where $r = (\lceil r/h \rceil - 1)h + r_0$, $1 \leqslant r_0 \leqslant h$,

$$f(q,m) = \frac{q^{mh+r_0}(q^h - 1)}{q^{mh+r_0} - 1}$$

for all m *such that* $0 \leqslant m \leqslant \lceil r/h \rceil - 2$.

Corollary of Bound

Corollary

If there is a $[n, r/h, d]_q^h$ *additive code then*

$$n \geqslant \lceil r/h \rceil + d - m + \left\lceil \sum_{j=1}^{m-2} \frac{d}{q^{jh}} \right\rceil ,$$

where $r = (k-1)h + r_0$, $k = \lceil r/h \rceil$, $1 \leqslant r_0 \leqslant h$,

$$q^{(m-2)h+r_0} < d \leqslant q^{(m-1)h+r_0} \leqslant q^r.$$

Additive Griesmer Bound 2.

Theorem (Ball, Lavrauw, P. 2024)

If there is a $[n, r/h, d]_q^h$ *additive code then*

$$n \geqslant d + \frac{q-1}{q^h - 1} \sum_{j=1}^{r-h} \lceil \frac{d}{q^j} \rceil.$$

Additive Griesmer Bound 2.

Theorem (Ball, Lavrauw, P. 2024)

If there is a $[n, r/h, d]_q^h$ *additive code then*

$$n \geqslant d + \frac{q-1}{q^h-1}\sum_{j=1}^{r-h}\lceil \frac{d}{q^j}\rceil.$$

Theorem

If d *is such that* $q^{(m-2)h+r_0} < d \leqslant q^{(m-1)h+r_0}$, *for some* $m \in \{2, \dots, \lceil r/h \rceil\}$ *and*

$$(\lceil r/h \rceil - m)(q^h - 1) \geqslant (r-h)(q-1) + q^h - q$$

then the first first is better than the second.

Additive MDS codes

Theorem (Singleton Bound)

For an $[n, k, d]_q$ *linear code* $k \leq n - d + 1$

Theorem (Huffman 2013)

If there is an $[n, r/h, d]_q^h$ *additive code then the Singleton bound can be reformulated as* $\lceil r/h \rceil \leqslant n - d + 1.$

A code is called maximum distance sepearable (MDS) when it meets the bound.

Additive MDS Bound

Theorem (Griesmer)

If there exists a $[n, k, d]_{q^h}$ *linear MDS code then* $n \leqslant k - 1 + q^h$

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Theorem (Ball, Lavrauw, P. 2024)

If there exists an $[n, r/h, d]_q^h$ *additive MDS code then*

$$n \leqslant \lceil r/h \rceil - 2 + q^h + \frac{q^h - 1}{q^{r_0} - 1},$$

where $r = (\lceil r/h \rceil - 1)h + r_0$, $1 \leqslant r_0 \leqslant h$,

Constructions of additive MDS codes

Example (Ball, Lavrauw, P 2024)

1. If r_0 divides h then there is a $[n, 1+(r_0/h), n-1]_{q^{r_0}}^{h/r_0}$ additive MDS code where

$$n = q^h + \frac{q^h - 1}{q^{r_0} - 1}.$$

2. There is a $[2^{h+1}, 2+(1/h), d]_2^h$ additive MDS code.

3. There are 6 inequivalent $[12, 2.5, 10]_3^2$ additive MDS codes.

Construction of additive Codes from Linear codes

Theorem (Ball, P. 2025)

If there is a linear $[n, k, d]_q$ code then there is a $[n, k/h, d_{\mathrm{add}}]_q^h$ additive code where

$$d_{\mathrm{add}} \geqslant \sum_{j=0}^{h-1} \lceil \frac{d}{q^j} \rceil.$$

- A generalization of a theorem by Guan et al.

[1] C. Guan, R. Li, Y.Liu and Z. Ma
Some quaternary additive codes outperform linear counterparts
IEEE Transactions on Information Theory, vol. 69, no. 11, pp. 7122-7131, Nov. 2023.

New Construction of additive codes

Theorem (Ball, P. 2025)

If $h \leqslant s$ *and* $t \geqslant 2$ *then there is a* $[q^{st}-1, \frac{st+s+1}{h}, d]_q^h$ *additive code where*

$$d \geqslant q^{st} - 1 - \frac{q^{st}-1}{q^s-1} q^{s-h}.$$

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If $h \leqslant s$ *and* $t \geqslant 2$ *then there is a* $[q^{st}-1, \frac{st+s+1}{h}, d]_q^h$ *additive code where*

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- The above theorem reaches the additive griesmer bound when $t = 2$ and $s = h$.
- The code gives a $[63, 5, 45]_2^2$ code, which performs better than any known linear codes.

Integral additive codes that outperform linear codes

- There exists a $[21, 3, 18]_3^2$ additive code (Mathon et al. 2003)
- Six examples for $q = 2$ and $h = 2$ (Guan et al. 2023)
- Examples for the following parameters when n is sufficiently large $[n, 4, d]_2^2$, $[n, 3, d]_3^2$, $[n, 3, d]_2^3$, $[n, 3, d]_4^2$, $[n, 5, d]_2^2$, and $[n, 3, d]_5^2$ (Kurz 2024)

Generalized Maximal Arc

Definition

A maximal arc $\mathcal{X}$ of degree t in $PG(2, q)$ is a set of points such that every line is incident to 0 or t points.

Definition

A generalized maximal arc of degree t is a set $\mathcal{X}$ of $h-1$ dimensional subspaces in $PG(kh-1, q)$ such that every hyperplane contains 0 or t members of $\mathcal{X}$.

Corollary

A generalized maximal arc of degree t over $PG(kh-1, q)$ is equivalent to a $[n, k, n-t]_q^h$ additive two weight code.