Cameron-Liebler sets of generators in polar spaces with rank $d > 3$

Morgan Rodgers

RPTU Kaiserslautern-Landau

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(joint work with Maarten De Boeck, Jozefien D'haeseleer, and Ferdinand Ihringer)

Overview

- A Cameron-Liebler set is a collection of subspaces in a finite projective or polar space having certain nice combinatorial properties.
- In general, they can be thought of as meeting some theoretical bound on containing as many intersecting spaces as possible.
- These objects have since been generalized to many other contexts, where they have important connections to the eigenspaces of distance regular graphs.

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Cameron–Liebler sets of k-spaces in projective space

Definition (R., Storme, Vansweevelt (2018), Blokhuis, De Boeck, D'haeseleer (2019))

A set of k-spaces $\mathcal{L}$ in $\mathrm{PG}(n,q)$ is called a *Cameron–Liebler k-class* (CL k-class) if its characteristic vector $\boldsymbol{c}$ lies in the row space of the point-k-space incidence matrix A.

We can assume that $n \geq 2k+1$ without loss of generality.

It can be shown that in this case $|\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}_q$ for some integer $0 \leq x \leq q^n + 1$ called the *parameter* of $\mathcal{L}$.

For projective spaces, Row $A = V_0 \oplus V_1$, where $V_0 = \langle \boldsymbol{j} \rangle$ and V_1 are the first two eigenspaces of the distance regular Grassmann graph $J_q(n+1, k+1)$ under the standard ordering.

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CL sets of k-spaces

This forces the number of pairwise nontrivially intersecting elements of $\mathcal{L}$ to be as large as possible in relation to $|\mathcal{L}|$, giving an equivalent definition.

Definition

A set of k-spaces $\mathcal{L}$ in $\mathrm{PG}(n, q)$ with characteristic vector $\boldsymbol{c}$ is a CL k-class if and only if there is some $x \in \mathbb{Q}$ such that every k-space π of $\mathrm{PG}(n, q)$ intersects nontrivially with

$$x\left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q q^{k^2+k}\right) + \left(\begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q q^{k^2+k} - 1\right) \boldsymbol{c}(\pi)$$

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This number of intersections comes from Hoffman's coclique bound applied to the disjointness relation, which is based on the minimal eigenvalue of this relation.

Theorem

If $\mathrm{PG}(n, q)$ admits a k-spread, then a set of k-spaces $\mathcal{L}$ is a CL k-class if and only if $\mathcal{L}$ shares some fixed number x of k-spaces with every spread of $\mathrm{PG}(n, q)$.

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CL sets in polar spaces

It also makes sense to define CL sets of generators in a finite classical polar space $\mathcal{P}$ in a similar way.

Definition (De Boeck, R., Storme, Švob (2019), De Boeck, D'haeseleer (2020))

Let $\mathcal{L}$ be a set of generators of a rank d polar space. Then $\mathcal{L}$ is a *(degree one) Cameron-Liebler set of generators* if the characteristic vector $\boldsymbol{c}$ lies in $V_0 \oplus V_1$, where $V_0 = \langle \boldsymbol{j} \rangle$ and V_1 are the first two eigenspaces of the distance regular dual polar graph under the standard ordering.

We have an integer $0 \leq x \leq q^{e+d-1} + 1$ for which $|\mathcal{L}| = x \prod_{i=0}^{d-2}(q^{e+i} + 1)$, called the *parameter* of the CL set.

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The *parameter* e of a rank d polar space is the value for which the number of generators through a $(d-1)$-space is given by $q^e + 1$.

polar space	e
$\mathcal{Q}^+(2d-1, q)$	0
$\mathcal{H}(2d-1, q)$	1/2
$\mathcal{W}(2d-1, q)$	1
$\mathcal{Q}(2d, q)$	1
$\mathcal{H}(2d, q)$	3/2
$\mathcal{Q}^-(2d+1, q)$	2

A construction for certain rank 4 polar spaces

Let $\mathcal{P}$ be a rank 4 polar space with parameter $e \leq 1$, having an embedded GQ $\mathcal{P}' \subseteq \mathcal{P}$ with parameter $e+1$.

This means we have either

- $\mathcal{Q}(4,q) \leq \mathcal{Q}^+(7,q)$,
- $\mathcal{H}(4,q) \leq \mathcal{H}(7,q)$ with q square, or
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Theorem (De Boeck, D'haeseleer, R. (2025))

Let $\mathcal{M}$ be an m-ovoid of $\mathcal{P}'$. The set $\mathcal{L}$ of generators of $\mathcal{P}$ meeting $\mathcal{P}'$ precisely in an element of $\mathcal{M}$ is a CL set of generators in $\mathcal{P}$ with parameter $mq^{e+1}(q-1)$.

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Proof sketch

To prove this construction actually gives a CL set, it suffices to show that for every generator $\pi \in \mathcal{P}$ we have that the number of elements of $\mathcal{L}$ meeting π in a line is given by

$$\begin{cases} mq^{e+1}(q-1) - 1 + q^e(q^2+q+1) & \text{if } \pi \in \mathcal{L}\,, \\ mq^{e+1}(q-1) & \text{if } \pi \notin \mathcal{L}\,. \end{cases}$$

We show this count by exploiting the fact that the perp of $\mathcal{P}'$ is a plane (of the ambient space) meeting $\mathcal{P}$ in a conic $\mathcal{C}$ so every generator of $\mathcal{P}$ containing a point of $\mathcal{C}$ meets $\mathcal{P}'$ in a line; the elements of $\mathcal{L}$ are all disjoint from $\mathcal{C}$.

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A construction for higher rank polar spaces

To generalize the construction, we take $\mathcal{P}$ to be a rank $d+2 \geq 4$ polar space with parameter $e \leq 1$, having an embedded rank d polar space $\mathcal{P}' \subseteq \mathcal{P}$ with parameter $e+1$.

This means we have either

- $\mathcal{Q}(2d, q) \leq \mathcal{Q}^{+}(2d+3, q)$,
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- $\mathcal{Q}^{-}(2d+1, q) \leq \mathcal{Q}(2d+4, q)$.

But what do we use in place of an m-ovoid in $\mathcal{P}'$?

We need a set $\mathcal{M}$ of $(d-1)$-spaces of $\mathcal{P}'$ that generalizes the properties of an m-ovoid in a generalized quadrangle.

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A generalization of m-ovoids

One obvious condition is that every generator of $\mathcal{P}'$ should contain a constant number m of elements of $\mathcal{M}$.

To get a CL set from $\mathcal{M}$ we need an additional regularity property:

For every $(d-1)$-space $\sigma \in \mathcal{P}'$, the number of elements σ_0 of $\mathcal{M}$ meeting σ in a $(d-2)$-space, where $\langle \sigma, \sigma_0 \rangle$ is not a generator of $\mathcal{P}'$, should be

$$\begin{cases} mq^{e+1}(q-1) & \text{if } \sigma \notin \mathcal{M}\,, \\ (m-1)q^{e+1}(q-1) + q^{e+2}\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q & \text{if } \sigma \in \mathcal{M}\,. \end{cases}$$

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Regular m-ovoids of $(d-1)$-spaces

Definition

A set $\mathcal{M}$ of $(d-1)$-spaces in a rank d polar space $\mathcal{P}$ meeting the above conditions is called a *regular m-ovoid of $(d-1)$-spaces*.

This gives us the following.

Theorem (De Boeck, D'haeseleer, R. (2025))

Let $\mathcal{P}' \subseteq \mathcal{P}$ as above, and let $\mathcal{M}$ be a regular m-ovoid of $(d-1)$-spaces in $\mathcal{P}'$. Take $\mathcal{L}$ to be the set of generators of $\mathcal{P}$ meeting $\mathcal{P}'$ precisely in an element of $\mathcal{M}$. Then $\mathcal{L}$ is a CL set of generators in $\mathcal{P}$ with parameter $mq^{e+1}(q-1)$.

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Examples from these constructions

We are able to get some examples in rank 4 polar spaces from this construction, coming from m-ovoids in $\mathcal{Q}(4,q)$, $\mathcal{H}(4,q)$, or $\mathcal{Q}^-(5,q)$.

- m-ovoids of $\mathcal{Q}^-(5,q)$ correspond to the well-studied hemisystems of $\mathcal{H}(3,q^2)$, which exist if and only if q is odd; thus we have CL sets of generators with parameter $\frac{q^2(q^2-1)}{2}$ in $\mathcal{Q}(8,q)$ for all odd q.
- There are many m-ovoids of $\mathcal{Q}(4,q)$, including several infinite families, giving many CL sets in $\mathcal{Q}^+(7,q)$.
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We are able to get some examples in rank 4 polar spaces from this construction, coming from m-ovoids in $\mathcal{Q}(4, q)$, $\mathcal{H}(4, q)$, or $\mathcal{Q}^-(5, q)$.

- m-ovoids of $\mathcal{Q}^-(5, q)$ correspond to the well-studied hemisystems of $\mathcal{H}(3, q^2)$, which exist if and only if q is odd; thus we have CL sets of generators with parameter $\frac{q^2(q^2-1)}{2}$ in $\mathcal{Q}(8, q)$ for all odd q.
- There are many m-ovoids of $\mathcal{Q}(4, q)$, including several infinite families, giving many CL sets in $\mathcal{Q}^+(7, q)$.
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Examples from these constructions

We also have some nontrivial examples in rank 5 polar spaces arising from our construction.

- We can take $\mathcal{Q}^-(5,q) \subseteq \mathcal{Q}(6,q) \subseteq \mathcal{Q}^+(9,q)$ (q odd). Take $\mathcal{O}$ to be a hemisystem of $\mathcal{Q}^-(5,q)$, and $\mathcal{M}$ to be the set of lines of $\mathcal{Q}(6,q)$ meeting the $\mathcal{Q}^-(5,q)$ in precisely a point of $\mathcal{O}$. Then our construction gives a CL set in $\mathcal{Q}^+(9,q)$ with parameter $\frac{q^2(q^2-1)}{2}$ (De Boeck, D'haeseleer, R. (2025)).
- Take $\mathcal{Q}(6,3^h) \subseteq \mathcal{Q}^+(9,3^h)$. There exists an ovoid $\mathcal{O}$ of $\mathcal{Q}(6,3^h)$; letting $\mathcal{M}$ be the set of lines of $\mathcal{Q}(6,3^h)$ meeting a point of $\mathcal{O}$, we obtain a CL set of $\mathcal{Q}^+(9,3^h)$ with parameter $q(q^2-1)$ (Ihringer, R. (2025)).

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Final remarks

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- For polar spaces, we really want a construction that will work for higher rank d - this would settle a conjecture of Ihringer.
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- OTHER constructions of CL sets of generators? More examples for spaces other than hyperbolic?

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