On the Reducibility of Minihypers and the Extension Problem for Arcs and Codes

Assia Rousseva
Sofia University

(joint work with Ivan Landjev and Leo Storme)

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1. Preliminaries

Definition. An $[n,k,d]_q$-**code** is a k-dimensional subspace $\mathcal{C}$ of $\mathbb{F}_q^n$ where d is the minimal Hamming weight of a non-zero codeword.

Definition. (n,w)-**arc** in $\mathrm{PG}(r,q)$: a multiset $\mathcal{K}$ with

1) $|\mathcal{K}| = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n,w)-**blocking set (minihyper)** in $\mathrm{PG}(r,q)$: a multiset $\mathcal{F}$ with

1) $|\mathcal{F}| = n$;

2) for every hyperplane H: $\mathcal{F}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{F}(H_0) = w$.

Theorem. 1. The following objects are equivalent:

(1) $[n, k, d]_q$ linear codes of full length and the maximal number of coordinate positions that are identical in all codewords is s;

(2) $(n, n-d)$-arcs in $\mathrm{PG}(k-1, q)$ with a maximal point multiplicity s;

(3) $(sv_k - n, sv_{k-1} - n + d)$-minihypers in $\mathrm{PG}(k-1, q)$.

Here: $v_k = \dfrac{q^k - 1}{q - 1}$.

Definition. An $[n, k, d]_q$-code with generator matrix G is called **t-extendable** if there exist t column-vectors $\boldsymbol{g}_1, \cdots, \boldsymbol{g}_t \in \mathbb{F}_q^k$ such that

$$[G|\boldsymbol{g}_1| \cdots |\boldsymbol{g}_t]$$

is a generator matrix of an $[n+t, k, d+t]_q$-code

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called **t-extendable** if there exists an $(n + t, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(r, q)$ with

$$\mathcal{K}'(P) \geq \mathcal{K}(P)$$

for all points P in $\mathrm{PG}(r, q)$.

Definition. An (n, w)-minihyper $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called **t-reducible** if there exists an $(n - t, w)$-minihyper $\mathcal{F}'$ in $\mathrm{PG}(r, q)$ with

$$\mathcal{F}'(P) \leq \mathcal{F}(P)$$

for all points P in $\mathrm{PG}(r, q)$.

t-extendable $[n, k, d]_q$-code

$$\Updownarrow$$

t-extendable $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$

$$\Updownarrow$$

t-reducible $(sv_k - n, sv_{k-1} - n + d)$-minihyper in $\mathrm{PG}(k-1, q)$

2. Classical Extension/Reducibility Results

- adding a parity-check bit

$$\exists \quad [n,k,d]_2, \quad d \text{ odd} \implies \exists \quad [n+1,k,d+1]_2$$

- maximal arcs

$$\exists \;\; ((n-1)(q+1),n)\text{-arc in } \mathrm{PG}(2,q) \implies \exists \;\; ((n-1)(q+1)+1,n)\text{-arc in } \mathrm{PG}(2,q)$$

Theorem. 2A. (R. Hill, P. Lizak, 1995) Every $[n,k,d]_q$-code with $\gcd(d,q)=1$ satisfying $A_i = 0$ for all $i \not\equiv 0, d \pmod q$ is extendable to an $[n+1,k,d+1]_q$-code.

Theorem. 2B. Let $\mathcal{K}$ be an (n,w)-arc in $\mathrm{PG}(r,q)$ with $w \equiv n+1 \pmod q$. Assume that the multiplicities of all hyperplanes are congruent to n or $n+1$ modulo q. Then $\mathcal{K}$ can be extended to an $(n+1,w)$-arc.

Theorem. 2C. Let $\mathcal{F}$ be an (n,w)-minihyper in $\mathrm{PG}(r,q)$, $w \equiv n-1 \pmod q$, such that the multiplicities of all hyperplanes are n or $n-1$ modulo q, then $\mathcal{F}$ can be reduced to an $(n-1,w)$-minihyper.

Theorem. 3A. (T. Maruta, T. Tanaka, H. Kanda, 2014) For $q = 2^h$ with $h \geq 3$. Every $[n, k, d]_q$-code with $\gcd(d, q) = 2$ satisfying $A_i = 0$ for all $i \not\equiv 0, d \pmod q$ is extendable to an $[n + 1, k, d + 1]_q$-code.

Theorem. 3B. Let $q = 2^h$ with $h \geq 3$. Every (n, w)-arc $\mathcal{K}$ with $\gcd(n - w, q) = 2$ satisfying $\mathcal{K}(H) \equiv w$ or $n \pmod q$ for all hyperplanes H is extendable to an $(n + 1, w)$-arc.

Theorem. 3C. Let $q = 2^h$ with $h \geq 3$. Every (n, w)-minihyper $\mathcal{F}$ with $\gcd(n - w, q) = 2$ satisfying $\mathcal{F}(H) \equiv n$ or $w \pmod q$ for all hyperplanes F is reducible to an $(n - 1, w)$-minihyper.

Theorem. 4A. (Maruta, 2001) Let $\mathcal{C}$ be an $[n,k,d]_q$-code, $q \geq 5$ odd, $d \equiv -2 \pmod q$, satisfying $A_i = 0$ for all $i \not\equiv 0,-1,-2 \pmod q$. Then $\mathcal{C}$ is extendable to an $[n+2,k,d+2]_q$-code.

Theorem. 4B. Let $\mathcal{K}$ be an (n,w)-arc in $\mathrm{PG}(r,q)$, $q \geq 5$ odd, with $w \equiv n+2 \pmod q$. Assume that the multiplicities of all hyperplanes are congruent to $n, n+1$ or $n+2$ modulo q. Then $\mathcal{K}$ is doubly extendable to an $(n+2,w)$-arc.

Theorem. 4C. Let $\mathcal{F}$ be an (n,w)-minihyper in $\mathrm{PG}(r,q)$, $q \geq 5$ odd, with $w \equiv n-2 \pmod q$, such that the multiplicities of all hyperplanes are $n, n-1$ or $n-2$ mod1.ulo q, then $\mathcal{F}$ can be doubly reduced to an $(n-2,w)$-minihyper.

Theorem. 5A. (Kanda, 2020) Let $\mathcal{C}$ be an $[n, k, d]_3$-code with $\gcd(d, 3) = 1$, satisfying that $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod 9$. Then $\mathcal{C}$ is extendable to an $[n + 1, k, d + 1]_3$-code.

Theorem. 5B. Let $\mathcal{K}$ be an (n, w)-arc in $\mathrm{PG}(r, 3)$ with $w \equiv n + 2 \pmod 9$ whose possible hyperplane multiplicities are all $n, n+1$, or $n+2 \pmod 9$. Then $\mathcal{K}$ is extendable to an $(n + 1, w)$-arc.

Theorem. 5C. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, 3)$ with $w \equiv n - 2 \pmod 9$ whose possible hyperplane multiplicities are all $n, n - 1$, or $n - 2 \pmod 9$. Then $\mathcal{F}$ is reducible to an $(n - 1, w)$-minihyper.

Theorem. 6A. (Kanda, 2022) Let $\mathcal{C}$ be an $[n,k,d]_4$-code with $k \geq 3, d \equiv -2 \pmod{16}$, satisfying that $A_i = 0$ for all $i \not\equiv 0, -2 \pmod{16}$. Then $\mathcal{C}$ is extendable to an $[n+1,k,d+1]_4$-code.

Theorem. 6B. Let $\mathcal{K}$ be an (n,w)-arc in $\mathrm{PG}(r,4)$ with $w \equiv n+2 \pmod{16}$ whose possible hyperplane multiplicities are all n, or $n+2 \pmod{16}$. Then $\mathcal{K}$ is extendable to an $(n+1,w)$-arc.

Theorem. 6C. Let $\mathcal{F}$ be an (n,w)-minihyper in $\mathrm{PG}(r,4)$ with $w \equiv n-2 \pmod{16}$ whose possible hyperplane multiplicities are all $n-2$, or $n \pmod{16}$. Then $\mathcal{F}$ is reducible to an $(n-1,w)$-minihyper.

3. A Modified Definition for Extendability/Reducibility

Definition. An (n, w)-minihyper $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called **reducible** if there exists a subspace S in $\mathrm{PG}(r, q)$, $\dim S = j \geq 0$, such that $\mathcal{F}$ can be represented as

$$\mathcal{F} = \mathcal{F}' + \chi_S,$$

where $\mathcal{F}'$ is a minihyper with parameters $(n - v_{j+1}, w - v_j)$, and χ_S is the characteristic function of S.

Remark. If $j = 0$ we get the classical definition of a reducible minihyper.

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called **extendable** if there exists a subspace S in $\mathrm{PG}(r, q)$, $\dim S = j \geq 0$, such that the arc $\mathcal{K}$ defined by

$$\mathcal{K}' = \mathcal{K} + \chi_S,$$

where χ_S is the characteristic function of S is an $(n + v_{j+1}, w + v_j)$-arc in $\mathrm{PG}(r, q)$.

Definition. An (n, w)-minihyper $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called **t-reducible** if there exist subspaces $S_1, \cdots, S_t$ in $\mathrm{PG}(r, q)$, $\dim S_i = s_i \geq 0$, such that $\mathcal{F}$ can be represented as

$$\mathcal{F} = \mathcal{F}' + \sum \chi_{S_i},$$

where $\mathcal{F}'$ is a minihyper with parameters $(n - \sum_{i=1}^{t} v_{s_i+1}, w - \sum_{i=1}^{t} v_{s_i})$, and χ_{S_i} is the characteristic function of S_i.

Remark. If $s_1 = \cdots = s_t = 0$ we get the classical definition of a t-reducible minihyper.

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called ***t*-extendable** if there exist subspaces $S_1, \cdots, S_t$ in $\mathrm{PG}(r, q)$, $\dim S_i = s_i \geq 0$, such that $\mathcal{K}'$ defined by

$$\mathcal{K}' = \mathcal{K} + \sum \chi_{S_i},$$

is an arc with parameters $(n + \sum_{i=1}^{t} v_{s_i+1}, w + \sum_{i=1}^{t} v_{s_i})$.

4. The Main Theorem

Definition. An (n, w)-arc (or an (n, w)-minihyper) $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called **divisible** with divisor Δ if

$$\mathcal{K}(H) \equiv n \pmod{\Delta}$$

for every hyperplane H.

Theorem. 7. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, $q = p^h$, with $w \equiv n - q^j \pmod{q^{j+1}}$, $j \geq 0$. Assume that $\mathcal{F}$ has the following properties:

(1) $\mathcal{F}(H) \equiv n - q^j$ or $n \pmod{q^{j+1}}$ for every hyperplane H in $\mathrm{PG}(r, q)$;

(2) for every hyperplane H with $\mathcal{F}(H) \equiv n - q^j \pmod{q^{j+1}}$, $\mathcal{F}|_H = \mathcal{F}_1 + \chi_T$ for a unique $(j-1)$-dimensional subspace T and $\mathcal{F}_1$ is a divisible minihyper with divisor q^j;

(3) for every hyperplane H with $\mathcal{F}(H) \equiv n \pmod{q^{j+1}}$, $\mathcal{F}|_H$ is a divisible minihyper with divisor q^j.

Then $\mathcal{F} = \mathcal{F}' + \chi_S$, where $\mathcal{F}'$ is an $(n - v_{j+1}, w - v_j)$-minihyper, and S is an j-dimensional subspace. In addition, the subspace S is uniquely determined.

Corollary. 8. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, with $w \equiv n - q \pmod{q^2}$. Assume that $\mathcal{F}$ has the following properties:

(1) $\mathcal{F}(H) \equiv n - q$ or $n \pmod{q^2}$ for every hyperplane H in $\mathrm{PG}(r, q)$;

(2) for every hyperplane H with $\mathcal{F}(H) \equiv n - q \pmod{q^2}$, $\mathcal{F}|_H$ is reducible to a divisible minihyper with divisor q;

(3) for every hyperplane H with $\mathcal{F}(H) \equiv n \pmod{q^2}$, $\mathcal{F}|_H$ is a divisible minihyper with divisor q.

Then $\mathcal{F} = \mathcal{F}' + \chi_L$, where $\mathcal{F}'$ is an $(n - q - 1, w - 1)$-minihyper, and L is a line.

5. Examples

Theorem. A $(70, 22)$-minihyper in $\mathrm{PG}(4, 3)$ is one of the following:

(1) the sum of a solid and a $(30, 9)$-minihyper in $\mathrm{PG}(4, 3)$;

(2) the sum of a $(66, 21)$-minihyper in and a line in $\mathrm{PG}(4, 3)$.

THANK YOU FOR YOUR ATTENTION!