Design switching on graphs

Robin Simoens

Ghent University & Universitat Politècnica de Catalunya

4 September 2025

Based on joint work with Ferdinand Ihringer (SUSTech)

Design switching on graphs

Definition

An (r, λ)-**design** is an incidence structure where

- every point is in r blocks,
- every two points are in λ blocks.

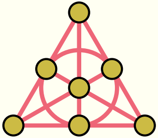

Figure: An $(r = 3, \lambda = 1)$-design

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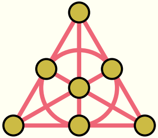

Figure: An $(r = 3, \lambda = 1)$-design

Definition

Switching is a local graph operation, resulting in a *cospectral graph*.

Cospectral graphs

Definition

Cospectral graphs have the same adjacency spectrum.

Figure: Cospectral graphs. Both have spectrum $\{-2, 0, 0, 0, 2\}$.

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Conjecture (van Dam and Haemers, 2003)

Almost all graphs are determined by their spectrum.

Cospectral graphs

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- Interesting for complexity theory

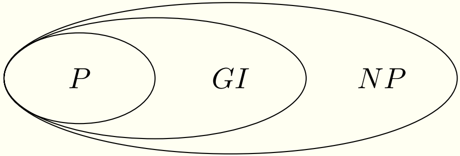

Figure: Is graph isomorphism an easy or hard problem?

Cospectral graphs

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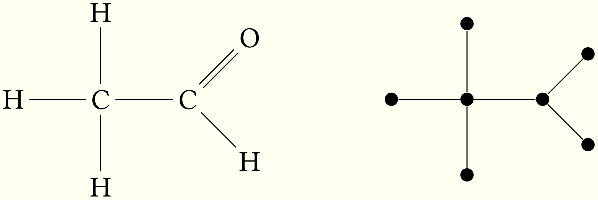

Figure: The molecular graph of acetaldehyde (ethanal).

Cospectral graphs

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🙂 Computational evidence [Brouwer and Spence, 2009]

n	3	4	5	6	7	8	9	10	11
ratio	1	1	0.941	0.936	0.895	0.861	0.814	0.787	0.789

Cospectral graphs

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- trees [Schwenk, 1973]
- strongly regular graphs [Fon-Der-Flaass, 2002]
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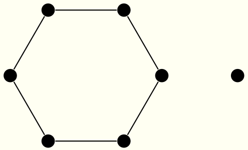
are **not** determined by their spectrum.

🙂 Exponentially many graphs are determined by their spectrum [Koval and Kwan, 2023]

How to find cospectral graphs

Theorem (GM$_4$ switching, Godsil and McKay, 1982)

Consider a graph with a regular subgraph C of size 4 such that every vertex $x \notin C$ has 0, 2 or 4 neighbours in C. If $x \notin C$ has 2 neighbours in C, reverse its adjacencies with C. The obtained graph is cospectral.



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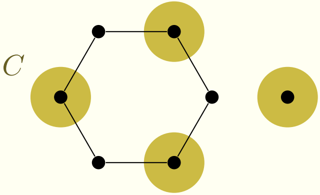

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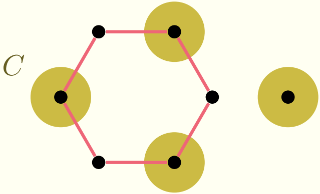

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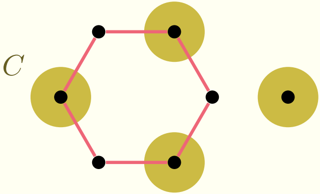

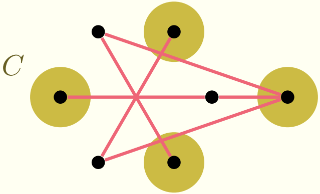

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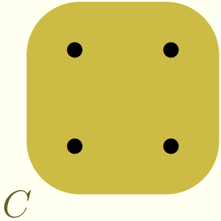
Proof.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}.$$

□

How to find cospectral graphs



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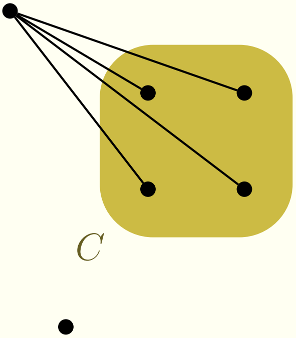

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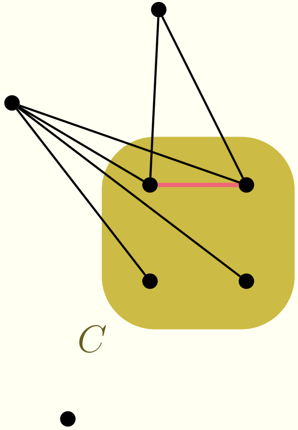
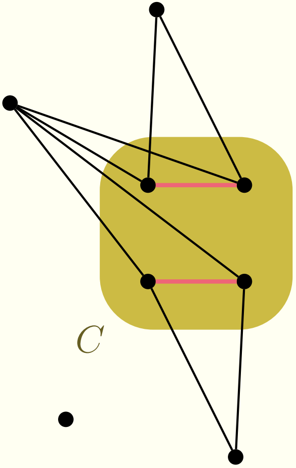
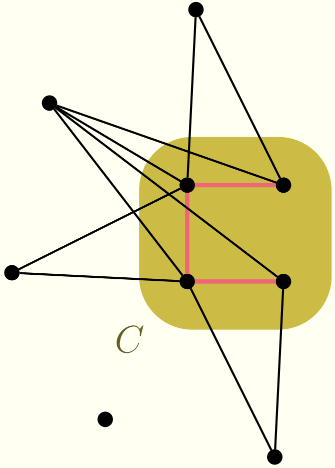



C

How to find cospectral graphs



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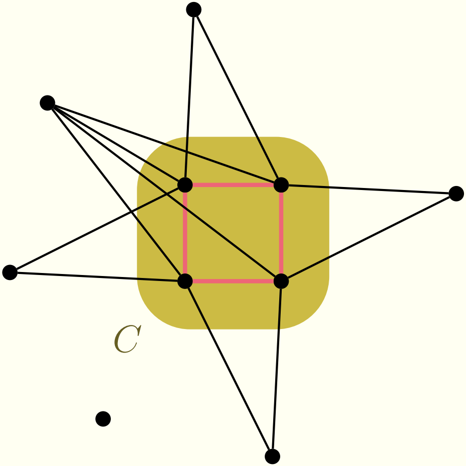

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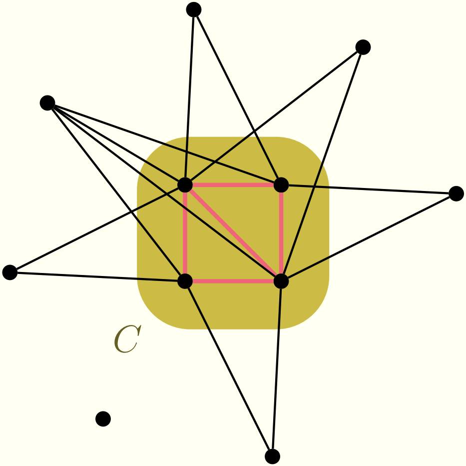

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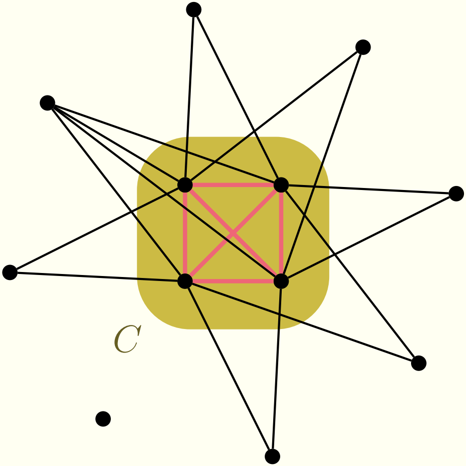

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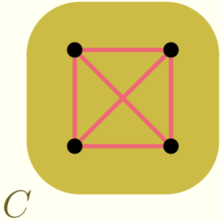
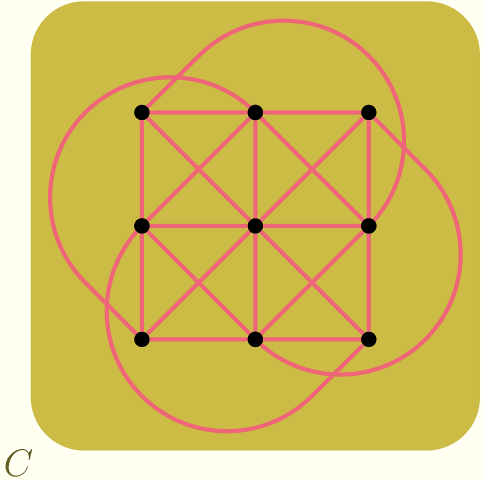
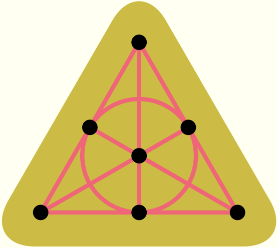


$\mathrm{AG}(2,2)$

How to find cospectral graphs



$\mathrm{AG}(2,3)$



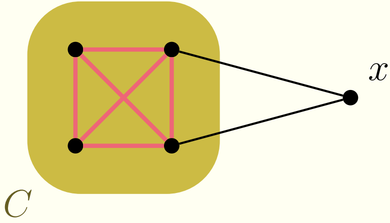
C

$\mathrm{PG}(2,2)$

Design switching

Theorem (Ihringer and Simoens, 2025+)

Consider a graph with a certain subgraph C whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block. Let π be a certain permutation of the blocks. If $x \notin C$ is adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The obtained graph is cospectral.



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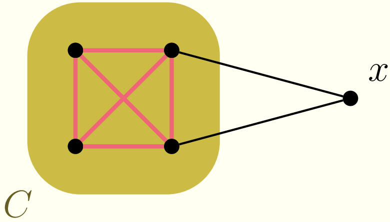

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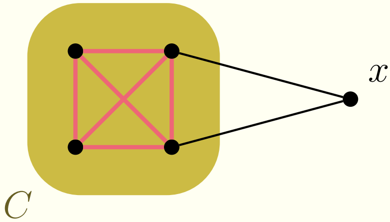


Known switching methods

Definition

Switching is a local graph operation, resulting in a *cospectral graph*.

- GM-switching [Godsil and McKay, 1982]
- WQH-switching [Wang, Qiu and Hu, 2019]
- AH-switching [Abiad and Haemers, 2012]
  - Sun graph switching [Mao, Wang, Liu and Qiu, 2023]
  - Fano switching [Abiad, van de Berg and Simoens, 2025+]
  - Cube switching [Abiad, van de Berg and Simoens, 2025+]

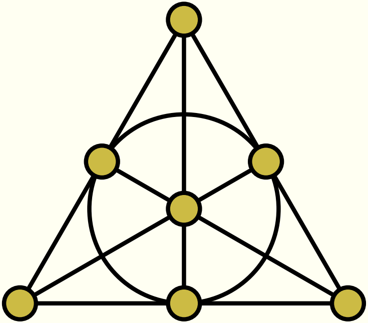
Fano switching

Abiad and Haemers (2012):

Conjugation of the adjacency matrix A with $Q = \begin{pmatrix} R & O \\ O & I \end{pmatrix}$, where

$$R = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Fano switching



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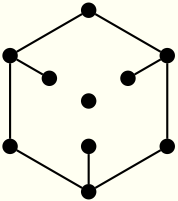
Fano switching

Theorem (Abiad, van de Berg and Simoens, 2025+)

Consider a graph with a subgraph C whose vertices are identified as points of the Fano plane such that:

- *C is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in C.*
  - *If x has 3 neighbours in C, they form a line.*
  - *If x has 4 neighbours in C, they form the complement of a line.*

Let π be a permutation of the lines. If $x \notin C$ is (non)adjacent to the vertices of ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The obtained graph is cospectral.



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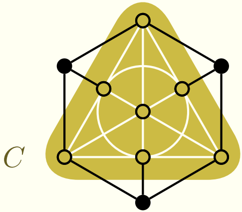

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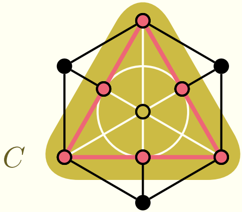

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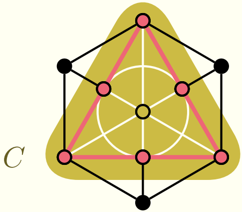

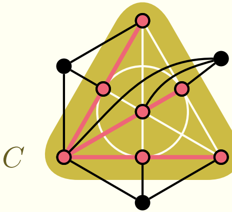

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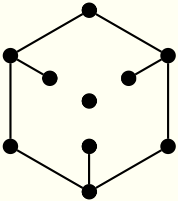

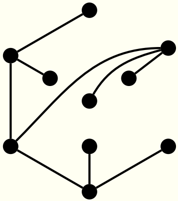

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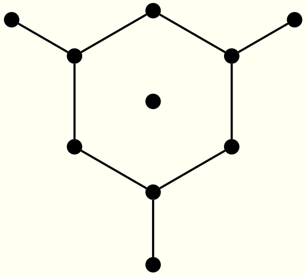

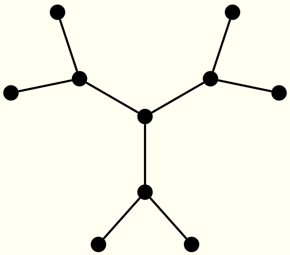

Both graphs have spectrum $\left\{(-\sqrt{5})^1, (-\sqrt{2})^2, (0)^3, (\sqrt{2})^2, (\sqrt{5})^1\right\}$.

Design switching

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Theorem (Ihringer and Simoens, 2025+)

Consider a graph with an edgeless or complete subgraph C whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block. Let π be a certain permutation of the blocks. If $x \notin C$ is adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The obtained graph is cospectral.

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$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)|.$$

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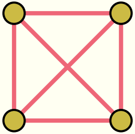

is an $(r = 3, \lambda = 1)$-design with incidence matrix

$$
\begin{array}{c}
 \\
p_1 \\
p_2 \\
p_3 \\
p_4
\end{array}
\begin{array}{c}
\begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
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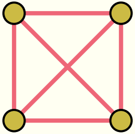
\end{pmatrix}
\end{array}
$$

Design switching



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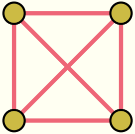
$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection



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\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}
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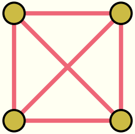

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$$\begin{array}{c|cccccc} & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ \hline p_1 & 1 & 1 & 1 & 0 & 0 & 0 \\ p_2 & 1 & 0 & 0 & 1 & 1 & 0 \\ p_3 & 0 & 1 & 0 & 1 & 0 & 1 \\ p_4 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

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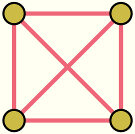

is an $(r = 3, \lambda = 1)$-design with incidence matrix

$$\begin{array}{c c} & \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection

Theorem (GM$_4$ switching, Godsil and McKay, 1982)

Consider a graph with a regular subgraph C of size 4 such that every vertex $x \notin C$ has $0, 2$ or 4 neighbours in C. If $x \notin C$ has 2 neighbours in C, reverse its adjacencies with C. The obtained graph is cospectral.

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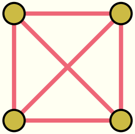

is an $(r = 4, \lambda = 2)$-design with incidence matrix

$$\begin{array}{cc} & \begin{array}{cccccccc} & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & \end{array} \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

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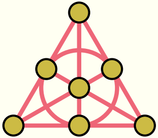

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$$
\begin{array}{c}
 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
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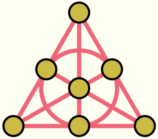
0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Any permutation of the lines π preserves pairwise intersection

Design switching



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$$
\begin{array}{c}
\\
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$$

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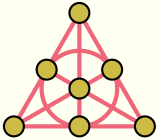
- Fano switching



is an $(r = 8, \lambda = 4)$-design with incidence matrix

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & \overline{B_1} & \overline{B_2} & \overline{B_3} & \overline{B_4} & \overline{B_5} & \overline{B_6} & \overline{B_7} & \end{array} \\
\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
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0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

Any permutation of the lines π preserves pairwise intersection

➤ Fano switching

Design switching

Theorem (Ihringer and Simoens, 2025+)

Consider a graph with an edgeless or complete subgraph C whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block. Let π be a permutation of the blocks such that for all blocks B_i, B_j,

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| \,.$$

If $x \notin C$ is adjacent to the points of B, make it adjacent to the points of $\pi(B)$. The obtained graph is cospectral.

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Proof. Define $R = \frac{1}{r-\lambda}\left(N(N^\pi)^T - \lambda J\right)$, where N^π is obtained from the incidence matrix N by permuting the columns with π.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} R & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R & O \\ O & I \end{pmatrix}.$$

□

Design switching

Theorem (Ihringer and Simoens, 2025+)

Consider a graph with a subgraph C with adjacency matrix $A_{11} = R^T A_{11} R$ whose vertices are identified as points of an (r,λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block. Let π be a permutation of the blocks such that for all blocks B_i, B_j,

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Consider a graph with a subgraph C with adjacency matrix $A_{11} = R^T A_{11} R$ whose vertices are identified as points of an (r, λ)-design such that every vertex $x \notin C$ is adjacent to the points of a block or its complement. Let π be a permutation of the blocks such that for all blocks B_i, B_j,

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□

Small 2-(v, k, λ)-designs

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v	# methods	Method
4	1	GM_4 switching
5	0	
6	1	GM_6 switching
7	1	Fano switching
8	10	AG$(3, 2)$-switching
9	≥ 2	AG$(2, 3)$-switching
10	≥ 4	
11	≥ 77	Paley biplane switching
12	≥ 6	
13	≥ 187	PG$(3, 2)$-switching

Table: Switching methods from small 2-(v, k, λ)-designs.

An application

An application

Definition

The **triangular graph** T_n has as vertices the 2-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if they intersect.

$T_n \cong L(K_n) \cong J(n, 2)$

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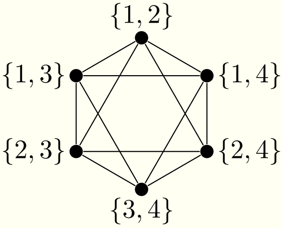

The octahedral graph T_4

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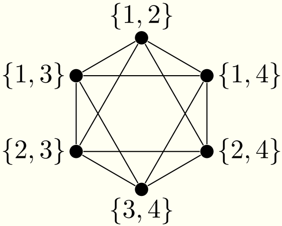

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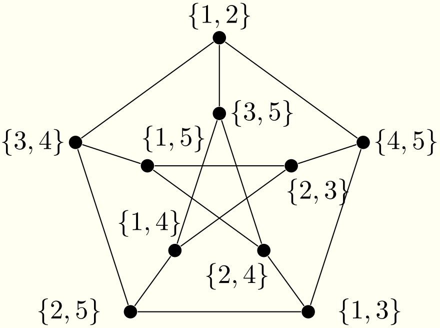

The Petersen graph $\overline{T_5}$

An application

Theorem (Chang and Hoffman, independently, 1959)

The triangular graph T_n is determined by its spectrum iff $n \neq 8$.

An application

Definition

The **q-triangular graph** $T_{q,n}$ has as vertices the 2-dimensional subspaces of $\mathbb{F}_q^n$ where two vertices are adjacent if they intersect.

An application

Definition

The **q-triangular graph** $T_{q,n}$ has as vertices the lines of $\mathrm{PG}(n-1, q)$ where two vertices are adjacent if they intersect.

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Theorem (Ihringer and Munemasa, 2019)

The q-triangular graph $T_{q,n}$ is not determined by its spectrum if $n \geq 4$.

Proof. Fix a subplane $\mathrm{PG}(2,q) \subseteq \mathrm{PG}(n-1,q)$ and let

$$\mathcal{P} = \{\text{lines of } \mathrm{PG}(2,q)\}$$
$$\mathcal{B} = \{\text{point pencils of } \mathrm{PG}(2,q)\}$$

Design switching on $(\mathcal{P},\mathcal{B})$, using any permutation π of $\mathcal{B}$ that is not an automorphism $\implies$ maximal cliques of size q^2+q. □

An application

Theorem (Ihringer and Simoens, 2025+)

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$$\Gamma_{\pi_1} \cong \Gamma_{\pi_2} \iff \pi_1, \pi_2 \in \text{ same double coset of } \mathrm{Aut}(D) \text{ in } \mathrm{Sym}(\mathcal{B})$$

There are $\geq q!$ double cosets. $\square$

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There are $\geq q!$ double cosets. $\square$

- Many strongly regular graphs with the same parameters.

Corollary (Fon-Der-Flaass, 2002)

*Almost all strongly regular graphs are **not** determined by their spectrum.*

Concluding remarks

- Many new switching methods

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- Alternative proofs of cospectrality results:
  - q-triangular graphs [Ihringer, Munemasa, 2019]
  - Collinearity graphs of polar spaces [Brouwer, Ihringer, Kantor, 2022]
  - Collinearity graphs of generalised quadrangles [Guo, van Dam, 2022]

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- More general: two different designs

Thank you for listening!

F. Ihringer and R. Simoens,
Design switching on graphs, `arXiv:2508.11523`.