Erdős-Ko-Rado sets on the hyperbolic quadrics $\mathcal{Q}^+(4n+1, q)$

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Irsee 2025

Outline

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HYPERBOLIC QUADRICS

- Hyperbolic quadric $\mathcal{Q}^+(2n+1,q)$:
  $X_0X_1 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0$,
- $\mathcal{Q}^+(2n+1,q)$ has points, lines, ..., n-spaces,
- n-spaces on $\mathcal{Q}^+(2n+1,q)$ are called *generators*,
- $\mathcal{Q}^+(2n+1,q)$ has $2(q+1)(q^2+1)\cdots(q^n+1)$ generators.

GENERATORS ON HYPERBOLIC QUADRIC

- Set of generators Ω on $\mathcal{Q}^+(2n+1,q)$ can be partitioned into two equivalence classes Ω_1 and Ω_2:

$$\Pi_1 \sim \Pi_2 \Leftrightarrow \dim(\Pi_1 \cap \Pi_2) \equiv n \pmod 2.$$

Hyperbolic quadric $\mathcal{Q}^+(3, q)$

Millenáris Park, Budapest

Generators on hyperbolic quadric

- Set of generators Ω on $\mathcal{Q}^+(2n+1,q)$ can be partitioned into two equivalence classes Ω_1 and Ω_2:

$$\Pi_1 \sim \Pi_2 \Leftrightarrow \dim(\Pi_1 \cap \Pi_2) \equiv n \pmod 2.$$

- Ω_1: **Latin generators**
  Ω_2: **Greek generators**
- $(q+1)(q^2+1)\cdots(q^n+1)$ Latin and Greek generators.
- For $\mathcal{Q}^+(4n+1,q)$, generators of Ω_1 pairwise intersect.
- For $\mathcal{Q}^+(4n+1,q)$, generators of Ω_2 pairwise intersect.

OUTLINE

P. Erdős

C. Ko

R. Rado

Erdős-Ko-Rado problem

Problem: What are largest sets of k-sets in n-set, pairwise intersecting in at least one element?

Theorem (Erdős-Ko-Rado)

If S is set of k-sets in n-set Ω, with $2k \leq n$, pairwise intersecting in at least one element, then $|S| \leq \binom{n-1}{k-1}$. If $2k+1 \leq n$, then equality only holds if S consists of all k-sets through fixed element of Ω.

$n = 2k$**:** If $n = 2k$, other sets with equality: all k-sets in fixed subset of size $n-1 = 2k-1$ of Ω.

q-ANALOG OF ERDŐS-KO-RADO PROBLEM

Problem: What are largest sets of $(k-1)$-subspaces in PG$(n-1, q)$, pairwise intersecting non-trivially?

THEOREM (HSIEH, FRANKL-WILSON, NEWMAN-GODSIL)

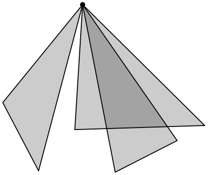
Let S be set of $(k-1)$-subspaces in PG$(n-1, q)$, with $2(k-1)+1 \leq n-1$, pairwise intersecting non-trivially.

- *If $2(k-1)+1 < n-1$, then largest examples consist of all $(k-1)$-spaces through fixed point.*
- *If $2(k-1)+1 = n-1$, then largest examples consist of all $(k-1)$-spaces through fixed point, or all $(k-1)$-spaces in given hyperplane of PG$(n-1=2k-1, q)$.*

Erdős-Ko-Rado problem in finite classical polar spaces

Problem:

- What are largest sets of generators in finite classical polar space P, pairwise intersecting non-trivially?
- All generators of P through fixed point (point-pencil = p.-p.).



V. Pepe and F. Vanhove

RESULTS FOR FINITE CLASSICAL POLAR SPACES

Polar space	Maximum size	Classification
$Q^-(2n+1,q)$	$(q^2+1)\cdots(q^n+1)$	p.-p.
$Q(4n,q)$	$(q+1)\cdots(q^{2n-1}+1)$	p.-p.
$Q(4n+2,q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$
$Q(6,q)$	$(q+1)(q^2+1)$	p.-p., Latins $Q^+(5,q)$, base
$Q^+(4n+1,q)$	$(q+1)\cdots(q^{2n}+1)$	all Latins
Latins $Q^+(4n+3,q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p.
Latins $Q^+(7,q)$	$(q+1)(q^2+1)$	p.-p., meeting Greek in plane
$W(4n+1,q)$, $n \geq 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	p.-p.
$W(4n+1,q)$, $n \geq 2$, q even	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$
$W(5,q)$, q odd	$(q+1)(q^2+1)$	p.-p., base,
$W(5,q)$, q even	$(q+1)(q^2+1)$	p.-p., base, Latins $Q^+(5,q)$
$W(4n+3,q)$	$(q+1)\cdots(q^{2n+1}+1)$	p.-p.
$H(2n,q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n-1}+1)$	p.-p.
$H(4n+3,q^2)$	$(q+1)(q^3+1)\cdots(q^{4n+1}+1)$	p.-p.
$H(4n+1,q^2)$, $n \geq 2$	$< \lvert\Omega\rvert/(q^{2n+1}+1)$	?
$H(5,q^2)$	$q(q^4+q^2+1)+1$	base

LARGEST EKR-SETS ON $\mathcal{Q}^+(4n+1, q)$

THEOREM

The largest Erdős-Ko-Rado sets on the hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$ are the set of Latin and the set of Greek generators of $Q^+(4n+1, q)$.

This example has size $(q+1)(q^2+1)\cdots(q^{2n}+1) \in O(q^{2n^2+n})$.
Comparison with point-pencil:
Point-pencil on $\mathcal{Q}^+(4n+1, q)$ has size

$$2(q+1)(q^2+1)\cdots(q^{2n-1}+1) \in O(q^{2n^2-n}).$$

CLASSIFICATION OF EKR SETS ON $\mathcal{Q}^+(5, q)$

THEOREM

There are three distinct maximal EKR-sets on the hyperbolic quadric $\mathcal{Q}^+(5, q)$:

1. *One class of generators.*
2. *Set consisting of one fixed generator Π and all generators of the other class intersecting this fixed generator in at least one point.*
3. *Point-pencil: all generators through a fixed point.*

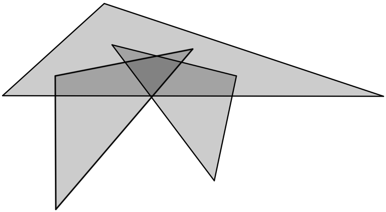
Second largest EKR-set on $\mathcal{Q}^+(4n+1,q)$

Theorem (De Boeck)

The second largest EKR-sets on $\mathcal{Q}^+(4n+1,q)$ are equal to the sets of generators consisting of one fixed generator Π, and the set of generators of the other class, intersecting this fixed generator Π in at least one point.

Class of generators: size $\in O(q^{2n^2+n})$.
Second largest example: size $\in O(q^{2n^2+n-1})$.

Second largest EKR-set on $Q^+(4n+1,q)$



THIRD LARGEST EKR-SET ON $\mathcal{Q}^+(4n+1,q)$

THEOREM (SCHELFHOUT AND STORME)

The third largest maximal EKR-sets on $\mathcal{Q}^+(4n+1,q)$ are constructed in the following way (up to interchange of Latin and Greek):

- *Consider the $q+1$ Latin generators Π_i, $i=1,\ldots,q+1$, through a fixed $(2n-2)$-dimensional space Ω.*
- *Consider the set of all Greek generators Π intersecting Ω in at least one point.*

The third largest maximal EKR-sets on $\mathcal{Q}^+(4n+1,q)$ are equal to the sets consisting of these $q+1$ Latin generators Π_i, $i=1,\ldots,q+1$, through a fixed $(2n-2)$-dimensional space Ω and these Greek generators intersecting in at least one point.

Size $\in O(q^{2n^2+n-2})$.

Steps in the proof

- There are at least two generators of each class.
- If all generators of one class pass through a common $(2n-2)$-dimensional space, then it the desired example.
- If $|E|$ is at least size of the desired example, then one class of generators contains at least $|E|/2$ generators of S.
- Assume that there are at least $|E|/2$ Greek generators.

Steps in the proof

- There are two Greek generators in E that share a $(2n-2)$-dimensional space π_{2n-2}.
- But not all Greek generators in E go through π_{2n-2}. So there is a third Greek generator in E not through π_{2n-2}.
- The number of Latin generators is at most

$$\prod_{i=1}^{2n}(q^i+1) - q^{2n^2+n} - q^{2n^2+n-1} - q^{2n^2+n-2}.$$

STEPS IN THE PROOF

- The number of Latin generators is at most

$$\prod_{i=1}^{2n}(q^i+1)-q^{2n^2+n}-q^{2n^2+n-1}-q^{2n^2+n-2}.$$

- No two Latin generators in E share a $(2n-2)$-dimensional space, or number of Greek generators is also at most

$$\prod_{i=1}^{2n}(q^i+1)-q^{2n^2+n}-q^{2n^2+n-1}-q^{2n^2+n-2}.$$

  .
  Then E is too small.

- So Latin generators in E pairwise intersect in at most $2n-4$ dimensions.

STEPS IN THE PROOF

- So Latin generators in E pairwise intersect in at most $2n-4$ dimensions.
- Then E is too small.
- Calculate size of third largest example minus upper bound on size of E leads to

$$q^{2n^2+n-3}-q^{2n^2+n-4}-(q^2+1)\prod_{i=4}^{2n}(q^i+1)+q+1>0.$$

Thank you very much for your attention