A lower bound on the minimum weight of some geometric codes

Rocco Trombetti

Department of Mathematics and Applications
University of Naples Federico II

joint work with: Bence Csajbók, Giovanni Longobardi and Giuseppe Marino

Finite Geometries 2025
Seventh Irsee Conference
31 August - 6 September 2025, Irsee, Germany

Overview

Codes arising from affine and projective spaces

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The code $\mathcal{C}_\Sigma(m, k, q)$ of points and k-spaces of Σ is the $\mathbb{F}_p$-span of the rows of A

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Minimum weight, minimum-weight codewords and p-rank of $\mathcal{C}$ are known

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$$\Updownarrow$$

$$v \in \mathcal{C}^\perp \Leftrightarrow Av^t = 0$$

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Assume $q > p$, and $c \in \mathcal{C}_{\mathrm{PG}}(2,q)^{\perp}$ has only coordinates 0 and 1; then

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A 0 mod p type multiset $\mathcal{M} \subset \Sigma$ is one such that for any line $\ell \subseteq \Sigma$ we have $|\mathcal{M} \cap \ell| \equiv 0$ mod p

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If $\mathcal{M}$ has at least one point x such that $\mu(x) = 1 \;\Rightarrow\; |\mathcal{M}| \geq 1 + (p-1)\frac{q^m-1}{q-1}$

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Let L be a $\mathbb{F}_p$-linear set of rank hm in $\mathrm{PG}(m,q)$, $q = p^h$. Then,

$$|L| \leq q^{m-1}\frac{q-1}{p-1} + \frac{q^{m-1}-1}{q-1}$$

A class of $0 \bmod p$ type multisets of $\mathrm{AG}(m, q)$ attaining the bound

- **Scattered polynomial**

Let $f = \sum_{i=0}^{r-1} a_i X^{p^i} \in \mathbb{F}_q[X]$ be an $\mathbb{F}_p$-linearized polynomial

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The polynomial f is said to be scattered if the following holds

$$\left| \left\{ \frac{f(x)}{x} : x \in \mathbb{F}_q^* \right\} \right| = \frac{q-1}{p-1}$$

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A lower bound on the minimum weight of $\mathcal{C}_{\mathrm{PG}}(m, q)^{\perp}$

Proposition (B. Csajbók, G. Longobardi, G. Marino, R.T.)

If $\mathcal{M} = (\mathcal{S}, \mu)$ is a $0 \bmod p$ type multiset of $\mathrm{PG}(m, q)$ meeting every hyperplane

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Theorem (B. Csajbók, G. Longobardi, G. Marino, R.T.)

Assume $q > p$, then

$$d_{\mathcal{C}_{PG}(m,q)^{\perp}} \geq 2(q^{m-1}(p-1)/p + q^{m-2})$$

A lower bound on the minimum weight of $\mathcal{C}_{\mathrm{PG}}(m, q)^{\perp}$

- **Sketch of the proof**

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  - If $p = 2 \quad \longrightarrow \quad$ Calkin, Key, De Resmini's bound

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- ○ scaling $\boldsymbol{s}$ does not change its weight $\Rightarrow$ we may assume a component of $\boldsymbol{s}$ equals 1

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- Since $|\mathcal{M}_{\boldsymbol{s}}|+|(p-1)\mathcal{M}_{\boldsymbol{s}})|=p|\mathcal{S}_{\boldsymbol{s}}|$

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- Since $|\mathcal{M}_{\boldsymbol{s}}| + |(p-1)\mathcal{M}_{\boldsymbol{s}})| = p|\mathcal{S}_{\boldsymbol{s}}| \Rightarrow 2((p-1)q^{m-1} + pq^{m-2}) \leq p|\mathcal{S}_{\boldsymbol{s}}|$

A lower bound on the minimum weight of $\mathcal{C}_{\mathrm{PG}}(m,q)^{\perp}$

Assume $\boldsymbol{s} \in \mathcal{C}_{\mathrm{PG}}(m,q)^{\perp}$

A lower bound on the minimum weight of $\mathcal{C}_{\mathrm{PG}}(m,q)^{\perp}$

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- If there is a hyperplane of $\mathrm{PG}(m,q)$ disjoint from $\mathcal{M}_{\boldsymbol{s}}$ $\longrightarrow$ apply previous arguments

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- **Concluding remarks**

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- **Concluding remarks**

  - For $m = 2$ we get Bagchi and Inamdar's bound

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- **Concluding remarks**

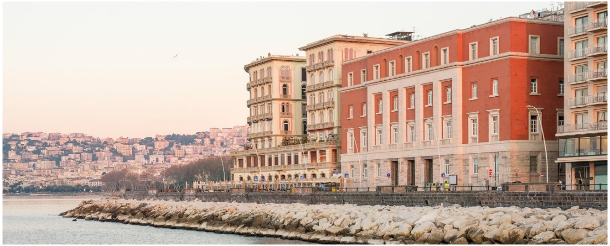
  - For $m = 2$ we get Bagchi and Inamdar's bound
  - For $q > p$ and $p, m > 2$ we improve on both Bagchi and Inamdar's and Lavrauw, Storme and Van de Voorde's bounds

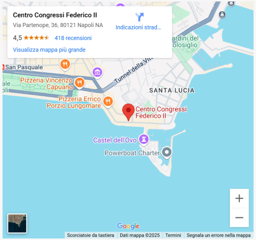
COMBINATORICS 2026

NAPLES, ITALY - MAY 25-29 2026

SPEAKERS

Anurag Bishnoi
Alain Couvreur
Tao Feng
Sam Mattheus
Gretchen L. Matthews
Maria Montanucci
Valentina Pepe
Martin Škoviera
Tommaso Traetta
Yue Zhou





Das Ende