The probability of two subspaces spanning a classical space

Geertrui Van de Voorde

University of Canterbury

Joint work with Maarten De Boeck

Finite Geometries (7th Irsee conference)
1–5 September 2025, Irsee, Germany.

Let's do some counting

- Given a point P in $\mathrm{PG}(2, q)$, how many lines are disjoint from P?

Let's do some counting

- Given a point P in $\mathrm{PG}(2, q)$, how many lines are disjoint from P?

There are $q^2 + q + 1 - (q + 1) = q^2$ lines disjoint from a given point

Let's do some counting

- Given a point P in $\mathrm{PG}(2, q)$, how many lines are disjoint from P?

There are $q^2 + q + 1 - (q + 1) = q^2$ lines disjoint from a given point (and this is almost all lines of $\mathrm{PG}(2, q)$).

Let's do some counting

- Given a point P in $\mathrm{PG}(2, q)$, how many lines are disjoint from P?

There are $q^2 + q + 1 - (q + 1) = q^2$ lines disjoint from a given point (and this is almost all lines of $\mathrm{PG}(2, q)$).

- Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

Solution 1: direct count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

Solution 1: direct count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- There are $(q^2 + 1)(q^2 + q + 1)$ lines in $\mathrm{PG}(3, q)$

Solution 1: direct count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- There are $(q^2 + 1)(q^2 + q + 1)$ lines in $\mathrm{PG}(3, q)$
- $1 + (q + 1)(q^2 + q)$ of those meet L.

Solution 1: direct count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- There are $(q^2 + 1)(q^2 + q + 1)$ lines in $\mathrm{PG}(3, q)$
- $1 + (q + 1)(q^2 + q)$ of those meet L.
- So, there are $(q^2 + 1)(q^2 + q + 1) - (q + 1)(q^2 + q) - 1 = q^4$ lines disjoint from L (and this is almost all lines of $\mathrm{PG}(3, q)$.)

Solution 2: look it up

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

Solution 2: look it up

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

Lemma (B. Segre 1961 – Lectures on Modern Geometry)

The number of $(j-1)$-dimensional subspaces of $\mathrm{PG}(n-1, q)$ disjoint from a given $(k-1)$-space equals

$$q^{kj} \begin{bmatrix} n-k \\ j \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}$.

Solution 2: look it up

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

Lemma (B. Segre 1961 – Lectures on Modern Geometry)

The number of $(j-1)$-dimensional subspaces of $\mathrm{PG}(n-1, q)$ disjoint from a given $(k-1)$-space equals

$$q^{kj} \begin{bmatrix} n-k \\ j \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\ldots(q-1)}$.

Here, $j = k = 2$, $n = 4$, so there are $q^{2\cdot 2}\begin{bmatrix} 2 \\ 2 \end{bmatrix}_q = q^4$ lines disjoint from a given line in $\mathrm{PG}(3, q)$.

Solution 3: double count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

SOLUTION 3: DOUBLE COUNT

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- Count pairs (M, π) where M is any line disjoint from L, and π is any plane through M.

Solution 3: double count

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- Count pairs (M, π) where M is any line disjoint from L, and π is any plane through M.
- We find $X \cdot (q + 1) =$

SOLUTION 3: DOUBLE COUNT

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- Count pairs (M, π) where M is any line disjoint from L, and π is any plane through M.
- We find $X \cdot (q + 1) =$

  (# planes π through L) $\cdot$ 0

  $+$ (# planes π not through L) $\cdot$ (# lines in π disjoint from $\pi \cap L$)

SOLUTION 3: DOUBLE COUNT

Given a line L in $\mathrm{PG}(3, q)$, how many lines are disjoint from L?

- Count pairs (M, π) where M is any line disjoint from L, and π is any plane through M.
- We find $X \cdot (q + 1) =$

  (# planes π through L) $\cdot\, 0$

  $+$ (# planes π not through L) $\cdot$ (# lines in π disjoint from $\pi \cap L$)

- Recursion

Motivation

Motivation

ELSEVIER

Finite Fields and Their Applications

Volume 82, September 2022, 102055

The probability of spanning a classical space by two non-degenerate subspaces of complementary dimensions

S.P. Glasby [a], Alice C. Niemeyer [a b], Cheryl E. Praeger [a]

The problem that we address in this paper arises from algorithmic considerations connected with computations in finite classical groups.

MOTIVATION

ELSEVIER

Finite Fields and Their Applications

Volume 82, September 2022, 102055

The probability of spanning a classical space by two non-degenerate subspaces of complementary dimensions

S.P. Glasby [a], Alice C. Niemeyer [a b], Cheryl E. Praeger [a]

The problem that we address in this paper arises from algorithmic considerations connected with computations in finite classical groups. In order to show that two isometries, each leaving invariant a non-degenerate proper subspace, generate a classical group with high probability a fundamental problem arises:

MOTIVATION

... show that, with high probability, for a vector space V endowed with a classical form,

Motivation

... show that, with high probability, for a vector space V endowed with a classical form, two random non-degenerate subspaces whose dimensions sum to $\dim(V)$, are complements of each other.

Recent results

Sample theorem from Glasby, Ihringer, Mattheus (2023):

Theorem 5.2 *Suppose* $V = (\mathbb{F}_{q^2})^{e_1+e_2}$ *is an* (e_1+e_2)*-dimensional hermitian space where* $e_1, e_2 \geq 1$. *For* $i \in \{1,2\}$, *let* Y_i *denote the set of all non-degenerate* e_i*-spaces of* V. *The proportion of pairs* $(S_1, S_2) \in Y_1 \times Y_2$ *for which* $S_1 \cap S_2 = \{0\}$ *is at least* $1 - \frac{c}{q^2}$ *where* $c = 2$ *when* $(e_1, e_2, q) = (1, 1, 2)$, $c = \frac{3}{2}$ *when* $\min\{e_1, e_2\} = 1$ *and* $(e_1, e_2, q) \neq (1, 1, 2)$, *and* $c = 1.26$ *otherwise.*

Recent results

Sample theorem from Glasby, Ihringer, Mattheus (2023):

Theorem 5.2 *Suppose $V = (\mathbb{F}_{q^2})^{e_1+e_2}$ is an (e_1+e_2)-dimensional hermitian space where $e_1, e_2 \geq 1$. For $i \in \{1,2\}$, let Y_i denote the set of all non-degenerate e_i-spaces of V. The proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is at least $1 - \frac{c}{q^2}$ where $c = 2$ when $(e_1, e_2, q) = (1,1,2)$, $c = \frac{3}{2}$ when $\min\{e_1, e_2\} = 1$ and $(e_1, e_2, q) \neq (1,1,2)$, and $c = 1.26$ otherwise.*

Similar results in

- Glasby S.P., Niemeyer A.C., Praeger C.E.: The probability of spanning a classical space by two non-degenerate subspaces of complementary dimensions. Finite Fields Their Appl. 82, 102055 (2022).
- Glasby S.P., Niemeyer A.C., Praeger C.E.: Random generation of direct sums of finite non-degenerate subspaces, Linear Algebra Appl. Linear Algebra Appl. 649, 408–432 (2022).

Motivation

Naive approach:

Motivation

Naive approach: the probability that a pair of two random subspaces is 'good' is

$$\frac{\text{number of 'good' pairs of subspaces}}{\text{total number of pairs of subspaces}}.$$

So...we should 'simply' count these quantities.

Anzahl theorems for formed spaces

- A formed space is a vector space together with a sesquilinear form.
- Comes in symmetric, symplectic, hermitian type ($\rightsquigarrow$orthogonal, symplectic, unitary groups)

Anzahl theorems for formed spaces

- A formed space is a vector space together with a sesquilinear form.
- Comes in symmetric, symplectic, hermitian type ($\rightsquigarrow$orthogonal, symplectic, unitary groups)
- Totally isotropic points define the classical polar spaces: quadrics, symplectic spaces, Hermitian varieties.

Anzahl theorems for formed spaces

- Given a line L in $\mathrm{PG}(3,q)$, how many secant (resp. passant) lines of a given $\mathcal{Q}^+(3,q)$ are disjoint from L?

Anzahl theorems for formed spaces

- Given a line L in $\mathrm{PG}(3, q)$, how many secant (resp. passant) lines of a given $\mathcal{Q}^+(3, q)$ are disjoint from L?
- Depends on L being secant, passant, tangent to $\mathcal{Q}^+(3, q)$

Anzahl theorems for formed spaces

- Given a line L in $\mathrm{PG}(3, q)$, how many secant (resp. passant) lines of a given $\mathcal{Q}^+(3, q)$ are disjoint from L?
- Depends on L being secant, passant, tangent to $\mathcal{Q}^+(3, q)$
- We can double count as before.
- But is there a known formula (for general dimension)?

Anzahl theorems for formed spaces

Anzahl theorems for formed spaces

- What is the number of points on a quadric or Hermitian variety?

Anzahl theorems for formed spaces

- What is the number of points on a quadric or Hermitian variety?

Wikipedia has the answer (non-singular forms)

Form	$n+1$	Name	Notation	Number of points	Collineation group
Alternating	$2r$	Symplectic	$W(2r-1,q)$	$(q^r+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma Sp}(2r,q)$
Hermitian	$2r$	Hermitian	$H(2r-1,q)$	$(q^{r-1/2}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma U}(2\mathrm{r},\mathrm{q})$
Hermitian	$2r+1$	Hermitian	$H(2r,q)$	$(q^{r+1/2}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma U}(2\mathrm{r}+1,\mathrm{q})$
Quadratic	$2r$	Hyperbolic	$Q^+(2r-1,q)$	$(q^{r-1}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}^+(2r,q)$
Quadratic	$2r+1$	Parabolic	$Q(2r,q)$	$(q^r+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}(2r+1,q)$
Quadratic	$2r+2$	Elliptic	$Q^-(2r+1,q)$	$(q^{r+1}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}^-(2r+2,q)$

Anzahl theorems for formed spaces

For subspaces of higher dimensions?

- An i-singular j-space corresponds to a cone with $(i-1)$-dimensional vertex and base a non-singular $(j-i-1)$ quadric/Hermitian variety/symplectic space.

Anzahl theorems for formed spaces

For subspaces of higher dimensions?

- An *i-singular j-space* corresponds to a cone with $(i-1)$-dimensional vertex and base a non-singular $(j-i-1)$ quadric/Hermitian variety/symplectic space.

Example: $n = 4$, $j = 3$, quadratic form of hyperbolic type

j-spaces correspond to planes meeting $\mathcal{Q}^+(3, q)$

Anzahl theorems for formed spaces

For subspaces of higher dimensions?

- An *i-singular j-space* corresponds to a cone with $(i-1)$-dimensional vertex and base a non-singular $(j-i-1)$ quadric/Hermitian variety/symplectic space.

Example: $n=4$, $j=3$, quadratic form of hyperbolic type

j-spaces correspond to planes meeting $\mathcal{Q}^+(3,q)$

- either 0-singular (plane meets in conic) or
- 1-singular (plane meets in 2 intersecting lines).

Anzahl theorems for formed spaces

- What is the number of *i*-singular *j*-spaces w.r.t. a form over a finite field?

Anzahl theorems for formed spaces

- What is the number of i-singular j-spaces w.r.t. a form over a finite field?

This is a classical result (see books of Hirschfeld or Wan).

Anzahl theorems for formed spaces

- What is the number of *i*-singular *j*-spaces w.r.t. a form over a finite field?

This is a classical result (see books of Hirschfeld or Wan).

Example theorem

Let $\alpha_{i,j,n}$ be the number of i-singular j-spaces w.r.t. a hermitian form on $\mathbb{F}_q^n$ (q square). For $0 \leq i \leq \min\{j, n-j\}$ and $j \leq n$ we have that

$$\alpha_{i,j,n} = q^{(j-i)(n-j-i)} \frac{\varphi^-_{j-i+1,n}(q)}{\varphi^-_{1,n-j-i}(q)\psi^-_{1,i}(q^2)} .$$

Here

$$\varphi^-_{a,b}(q) = \prod_{k=a}^{b} \left(q^k - (-1)^k\right) \quad \textit{and} \quad \psi^-_{a,b}(q) = \prod_{k=a}^{b} \left(q^k - 1\right)$$

BACK TO THE BEGINNING

THE GOAL

Given a non-degenerate polar space in $\mathrm{PG}(n, q)$

Back to the beginning

The goal

Given a non-degenerate polar space in $\mathrm{PG}(n, q)$ find the proportion of pairs (U, V) where U, V are disjoint non-singular subspaces of dimensions j and $n - j$

Back to the beginning

The goal

Given a non-degenerate polar space in $\mathrm{PG}(n, q)$ find the proportion of pairs (U, V) where U, V are disjoint non-singular subspaces of dimensions j and $n - j$ among all pairs of non-singular subspaces of dimensions j and $n - j$.

Back to the beginning

The goal

Given a non-degenerate polar space in $\mathrm{PG}(n, q)$ find the proportion of pairs (U, V) where U, V are disjoint non-singular subspaces of dimensions j and $n - j$ among all pairs of non-singular subspaces of dimensions j and $n - j$.

By transitivity

We can fix U and look for the number of V's disjoint from U.

THE FORMULA

Given π, a non-singular $(j-1)$-space, count pairs (τ, σ)
with τ a non-singular $(n-j)$-space disjoint from π
σ a non-singular hyperplane through τ.

THE FORMULA

Given π, a non-singular $(j-1)$-space, count pairs (τ, σ)
with τ a non-singular $(n-j)$-space disjoint from π
σ a non-singular hyperplane through τ.

DEFINITION

Consider a non-degenerate hermitian form f on $\mathbb{F}_{q^2}^n$.

- $\alpha_{i,j,n}$ is the number of i-singular j-spaces π.

The formula

Given π, a non-singular $(j-1)$-space, count pairs (τ, σ)
with τ a non-singular $(n-j)$-space disjoint from π
σ a non-singular hyperplane through τ.

Definition

Consider a non-degenerate hermitian form f on $\mathbb{F}_{q^2}^n$.

- $\alpha_{i,j,n}$ is the number of i-singular j-spaces π.
- $\beta_{i,j,n}$ is the number of non-singular hyperplanes $\sigma \supseteq \pi$ in $\mathbb{F}_{q^2}^n$.

The formula

Given π, a non-singular $(j-1)$-space, count pairs (τ, σ)
with τ a non-singular $(n-j)$-space disjoint from π
σ a non-singular hyperplane through τ.

Definition

Consider a non-degenerate hermitian form f on $\mathbb{F}_{q^2}^n$.

- $\alpha_{i,j,n}$ is the number of i-singular j-spaces π.
- $\beta_{i,j,n}$ is the number of non-singular hyperplanes $\sigma \supseteq \pi$ in $\mathbb{F}_{q^2}^n$.
- $\gamma_{i,j,n}$ is the number of non-singular $n-j$-spaces σ in $\mathbb{F}_{q^2}^n$ such that $\sigma \cap \pi$ is trivial.

The formula

Given π, a non-singular $(j-1)$-space, count pairs (τ, σ)
with τ a non-singular $(n-j)$-space disjoint from π
σ a non-singular hyperplane through τ.

Definition

Consider a non-degenerate hermitian form f on $\mathbb{F}_{q^2}^n$.

- $\alpha_{i,j,n}$ is the number of i-singular j-spaces π.
- $\beta_{i,j,n}$ is the number of non-singular hyperplanes $\sigma \supseteq \pi$ in $\mathbb{F}_{q^2}^n$.
- $\gamma_{i,j,n}$ is the number of non-singular $n-j$-spaces σ in $\mathbb{F}_{q^2}^n$ such that $\sigma \cap \pi$ is trivial.

We have a formula for α. β follows from elementary double counting.The goal is a formula for $\gamma_{0,j,n}$.

The formula

Not so elementary double counting

We count the tuples (σ, τ) with σ a non-singular hyperplane, and $\tau \subseteq \sigma$ a non-singular $(n-j)$-space disjoint from π.

The formula

Not so elementary double counting

We count the tuples (σ, τ) with σ a non-singular hyperplane, and $\tau \subseteq \sigma$ a non-singular $(n-j)$-space disjoint from π.

$$\begin{aligned}
\gamma_{i,j,n}\beta_{0,n-j,n} &= \alpha_{1,j-i-1,j-i}\left(\beta_{i+1,j-1,n} - \beta_{i,j,n}\right)\gamma_{i+1,j-1,n-1} \\
&\quad + \alpha_{0,j-i-1,j-i}\left(\beta_{i,j-1,n} - \beta_{i,j,n}\right)\gamma_{i,j-1,n-1} \\
&\quad + \left(\begin{bmatrix} j \\ j-1 \end{bmatrix}_{q^2} - \begin{bmatrix} j-i \\ j-i-1 \end{bmatrix}_{q^2}\right)\left(\beta_{i-1,j-1,n} - \beta_{i,j,n}\right)\gamma_{i-1,j-1,n-1}
\end{aligned}$$

The method summarised

- Derive a recursion formula for γ
- 'Guess' the closed formula
- Prove the formula by induction
- Main work was for complementary case, the general formula follows by another double counting argument.

The method summarised

- Derive a recursion formula for γ
- 'Guess' the closed formula
- Prove the formula by induction
- Main work was for complementary case, the general formula follows by another double counting argument.

The motivation was to find a formula for the 0-singular case but this method requires us to derive a formula for all *i*-singular cases at once.

The results (unitary case)

Theorem 3.7. *For* $0 \le i \le \min\{j, n-j\}$, $j \le n-1$ *and* $k \le n-j$, *we have that*

$$\begin{aligned}\gamma_{i,j,n,k} &= \beta_{i,j,n,k+j}\gamma_{i,j,k+j}\\ &= q^{(n-k-j)(k+i)+2jk-\binom{j+1}{2}} \begin{bmatrix} n-j-i \\ n-k-j \end{bmatrix}_q^- \varphi^+_{1,i}(q) \sum_{m=0}^{j-i} (-1)^{mk} \varphi^+_{i+1,j-m}(q) \begin{bmatrix} j-i \\ m \end{bmatrix}_q^- q^{\binom{m}{2}-m(k-i)}\end{aligned}$$

THE RESULTS (UNITARY CASE)

Theorem 3.7. *For $0 \le i \le \min\{j, n-j\}$, $j \le n-1$ and $k \le n-j$, we have that*

$$\gamma_{i,j,n,k} = \beta_{i,j,n,k+j}\gamma_{i,j,k+j}$$
$$= q^{(n-k-j)(k+i)+2jk-\binom{j+1}{2}} \begin{bmatrix} n-j-i \\ n-k-j \end{bmatrix}_q^- \varphi^+_{1,i}(q) \sum_{m=0}^{j-i} (-1)^{mk} \varphi^+_{i+1,j-m}(q) \begin{bmatrix} j-i \\ m \end{bmatrix}_q^- q^{\binom{m}{2}-m(k-i)}$$

- *The formula for* the exact proportion of good subspaces now follows directly.

Theorem 3.9. *For integers j, k, n with $0 \le j, k \le n-1$ and $j+k \le n$, we have*

$$\rho_{j,k,n} = q^{jk-\binom{j+1}{2}} \frac{\varphi^-_{n-j-k+1,n-k}(q)}{\varphi^-_{n-j+1,n}(q)} \sum_{m=0}^{j} (-1)^{mk} \varphi^+_{1,j-m}(q) \begin{bmatrix} j \\ m \end{bmatrix}_q^- q^{\binom{m}{2}-mk}$$

THE RESULTS (UNITARY CASE)

Theorem 3.7. *For $0 \leq i \leq \min\{j, n-j\}$, $j \leq n-1$ and $k \leq n-j$, we have that*

$$\gamma_{i,j,n,k} = \beta_{i,j,n,k+j}\gamma_{i,j,k+j}$$
$$= q^{(n-k-j)(k+i)+2jk-\binom{j+1}{2}} \begin{bmatrix} n-j-i \\ n-k-j \end{bmatrix}_q^- \varphi_{1,i}^+(q) \sum_{m=0}^{j-i} (-1)^{mk} \varphi_{i+1,j-m}^+(q) \begin{bmatrix} j-i \\ m \end{bmatrix}_q^- q^{\binom{m}{2}-m(k-i)}$$

- *The formula for* the exact proportion of good subspaces now follows directly.

Theorem 3.9. *For integers j, k, n with $0 \leq j, k \leq n-1$ and $j+k \leq n$, we have*

$$\rho_{j,k,n} = q^{jk-\binom{j+1}{2}} \frac{\varphi_{n-j-k+1,n-k}^-(q)}{\varphi_{n-j+1,n}^-(q)} \sum_{m=0}^{j} (-1)^{mk} \varphi_{1,j-m}^+(q) \begin{bmatrix} j \\ m \end{bmatrix}_q^- q^{\binom{m}{2}-mk}$$

REMARK

Getting from the exact proportion to a bound of the form $1 - c\frac{1}{q^2}$ is still *a lot* of work.

The other cases

- Unitary case forms the blueprint of the main idea

The other cases

- Unitary case forms the blueprint of the main idea
- Symplectic case: a bit more complicated because the need to count with subspaces of codimension 2

The other cases

- Unitary case forms the blueprint of the main idea
- Symplectic case: a bit more complicated because the need to count with subspaces of codimension 2
- Orthogonal case, odd characteristic: *much* more complicated because of elliptic, parabolic, and hyperbolic types, and the fact that the group can have more than one orbit on *i*-singular spaces.

THE OTHER CASES

- Unitary case forms the blueprint of the main idea
- Symplectic case: a bit more complicated because the need to count with subspaces of codimension 2
- Orthogonal case, odd characteristic: *much* more complicated because of elliptic, parabolic, and hyperbolic types, and the fact that the group can have more than one orbit on i-singular spaces. There are four recursion formulae depending on the parity of i, j, n so four different formula for $\gamma_{(i,j,\delta,\lambda),(n,\epsilon),(n-j,\zeta)}$.

THE OTHER CASES

SAMPLE THEOREM IN ORTHOGONAL CASE (DE BOECK-VDV)

Theorem 4.19. *Let n, j, i be integers with $0 \leq i \leq j$ and $i + j \leq n$, such that n is even and j is odd. Let $\delta \in \{0, \pm 1\}$ and $\varepsilon, \lambda \in \{\pm 1\}$ with $j - i - \delta \equiv 1 \pmod 2$. If i is odd, then*

$$\gamma_{(i,j,\delta),(n,\varepsilon),(n-j,0),\varepsilon} = q^{\frac{3}{2}jn - \frac{1}{4}n^2 - \frac{5}{4}j^2 - \frac{1}{2}n + \frac{1}{2}j - \frac{1}{4}} \left(\sum_{m=0}^{\frac{n-j-i}{2}} \chi_{1,\frac{n-j+1}{2}-m}(q) \begin{bmatrix} \frac{n-j-i}{2} - 1 \\ m \end{bmatrix}_{q^2} q^{m(m-j+i+1)} \right.$$
$$\left. + \left(\delta q^{\frac{1}{2}i - \frac{1}{2}j} - \varepsilon q^{\frac{1}{2}n-j} + \delta\varepsilon q^{\frac{1}{2}n - \frac{3}{2}j + \frac{1}{2}i} \right) \sum_{m=0}^{\frac{n-j-i}{2}-1} \chi_{1,\frac{n-j-1}{2}-m}(q) \begin{bmatrix} \frac{n-j-i}{2} - 1 \\ m \end{bmatrix}_{q^2} q^{m(m-j+i+1)}. \right)$$

If i is even, then

$$\gamma_{(i,j,0),(n,\varepsilon),(n-j,0),\varepsilon} = q^{\frac{3}{2}jn - \frac{1}{4}n^2 - \frac{5}{4}j^2 - \frac{1}{2}n + \frac{1}{2}j - \frac{1}{4}} \left(\sum_{m=0}^{\frac{n-j-i-1}{2}} \chi_{1,\frac{n-j+1}{2}-m}(q) \begin{bmatrix} \frac{n-j-i-1}{2} \\ m \end{bmatrix}_{q^2} q^{m(m-j+i)} \right.$$
$$\left. - \varepsilon q^{\frac{1}{2}n-j} \sum_{m=0}^{\frac{n-j-i-1}{2}} \chi_{1,\frac{n-j-1}{2}-m}(q) \begin{bmatrix} \frac{n-j-i-1}{2} \\ m \end{bmatrix}_{q^2} q^{m(m-j+i)} \right)$$

Some concluding remarks

- GAP was used to verify small cases (for fixed q)

Some concluding remarks

- GAP was used to verify small cases (for fixed q)
- Maple came in handy to help guess the closed formula (for general q)

Some concluding remarks

- GAP was used to verify small cases (for fixed q)
- Maple came in handy to help guess the closed formula (for general q)
- We prove that our solution satisfies the recursion by induction.

SOME CONCLUDING REMARKS

- GAP was used to verify small cases (for fixed q)
- Maple came in handy to help guess the closed formula (for general q)
- We prove that our solution satisfies the recursion by induction.
- Unfortunately, we didn't manage to get Maple/Sage to actually do the induction proof, partially because of 'tricks' like this:

Lemma 2.10. *For integers $b \geq a \geq 0$ we have*

$$\begin{bmatrix} b \\ a \end{bmatrix}_q = q^a \begin{bmatrix} b-1 \\ a \end{bmatrix}_q + \begin{bmatrix} b-1 \\ a-1 \end{bmatrix}_q \qquad \text{and} \qquad \begin{bmatrix} b \\ a \end{bmatrix}_q = \begin{bmatrix} b-1 \\ a \end{bmatrix}_q + q^{b-a} \begin{bmatrix} b-1 \\ a-1 \end{bmatrix}_q .$$

Some concluding remarks

- GAP was used to verify small cases (for fixed q)
- Maple came in handy to help guess the closed formula (for general q)
- We prove that our solution satisfies the recursion by induction.
- Unfortunately, we didn't manage to get Maple/Sage to actually do the induction proof, partially because of 'tricks' like this:

Lemma 2.10. *For integers $b \geq a \geq 0$ we have*

$$\begin{bmatrix} b \\ a \end{bmatrix}_q = q^a \begin{bmatrix} b-1 \\ a \end{bmatrix}_q + \begin{bmatrix} b-1 \\ a-1 \end{bmatrix}_q \quad \textit{and} \quad \begin{bmatrix} b \\ a \end{bmatrix}_q = \begin{bmatrix} b-1 \\ a \end{bmatrix}_q + q^{b-a} \begin{bmatrix} b-1 \\ a-1 \end{bmatrix}_q .$$

- Orthogonal case, even characteristic: six different recursion formulae...

Thank you!

M. De Boeck and G. Van de Voorde. Anzahl theorems for trivially intersecting subspaces generating a non-singular subspace I: symplectic and hermitian forms. *Linear Algebra Appl.* **699** (2024), 367–402.

M. De Boeck and G. Van de Voorde. Anzahl theorems for disjoint subspaces generating a non-degenerate subspace: quadratic forms. *Combinatorial Theory* (2025), **5 (2)**, # 12.

EXAMPLE: $i = 0$, $j = 3$, $k = 2$, $\zeta = \lambda = 1$ AND $n = 5$ (SO $\delta = \epsilon = \eta = 0$)

Then we are looking for the proportion of pairs of conic planes of perp type 1 and secant lines to a parabolic quadric $\mathcal{Q}(4, q)$ which span the entire space, among all such pairs. This proportion is

$$\begin{aligned}\rho_{(3,0,1),(2,1),(5,0),0} &= \frac{\gamma_{(0,3,0,1),(5,0),(2,1),0}}{\alpha_{(0,2,1),(5,0)}} \\ &= 1 - \frac{1}{q}\frac{q^3 + 2q^2 + q - 2}{(q+1)\left(q^2+1\right)} .\end{aligned}$$

So $\rho_{(3,0,1),(2,1),(5,0),0} > 1 - \frac{23}{20}\frac{1}{q}$ for $q \geq 3$.