Group testing via residuation and partial geometries[1]

Finite Geometries 2025 - Seventh Irsee Conference

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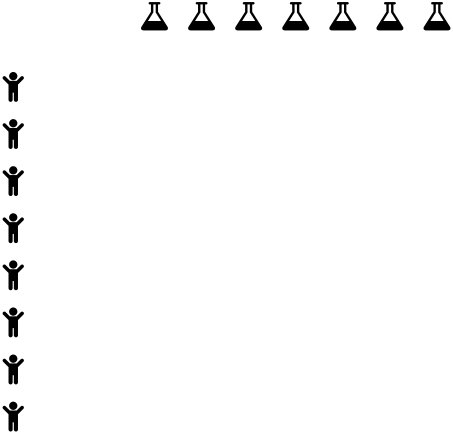

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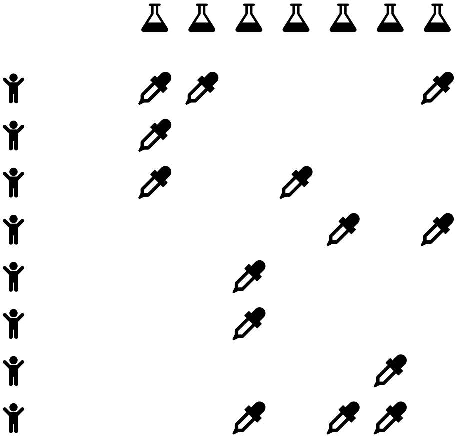

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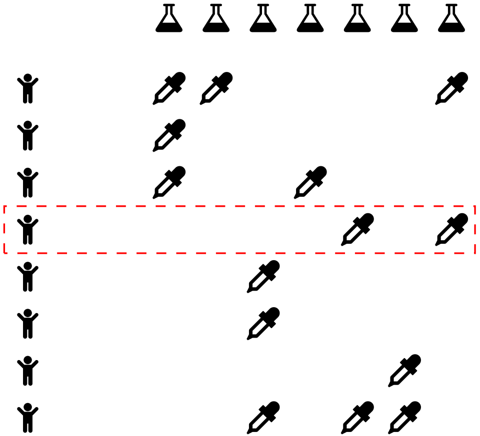

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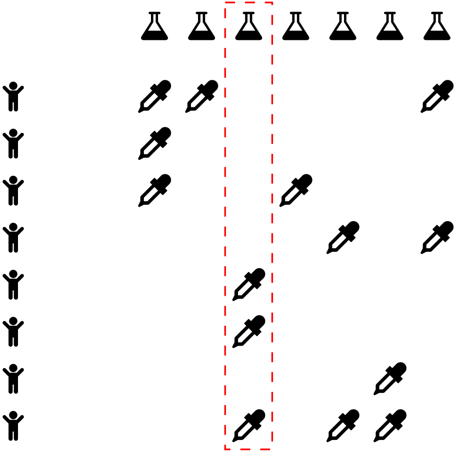

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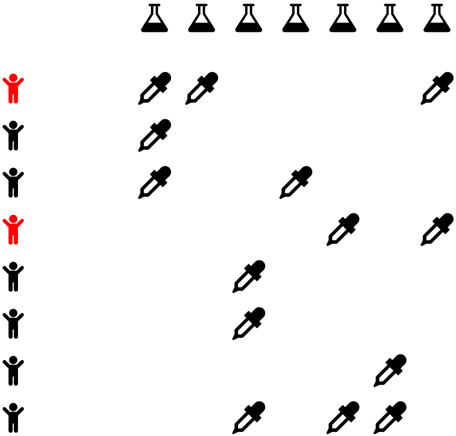

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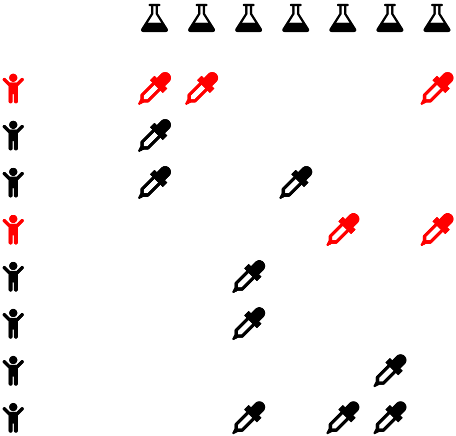

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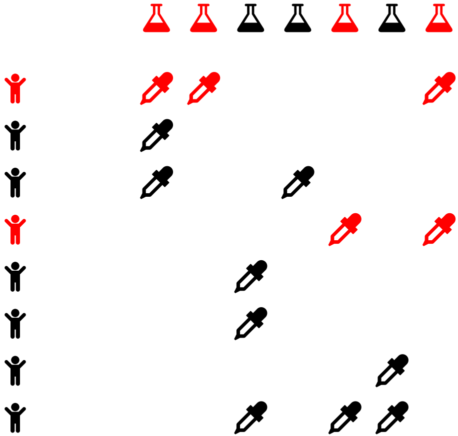

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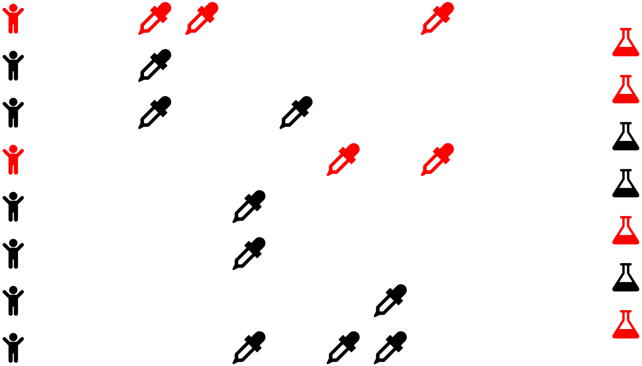

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$$
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
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\end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}^\top
$$

The problem of testing n items using k tests is modelled by a Boolean $n \times k$ matrix H by $x^\top H = y^\top$, where x denotes the items, H the test allocations, y the outcomes.

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Linear Algebra on $\mathbb{B}_2$

Let $\mathbb{B}_2$ denote the Boolean semiring ($\mathbb{F}_2$ but $1 + 1 = 1$), and $\mathbb{B}_2^n$ the semimodule of n-tuples over $\mathbb{B}_2$ with component-wise operations from $\mathbb{B}_2$. For $x, y \in \mathbb{B}_2^n$ we impose the ordering

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Lastly, $\overline{x}$ denotes the component-wise negation on x, satisfying De Morgan's laws.

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Let $H \in \mathbb{B}_2^{n \times k}$ and $d \in \mathbb{N}$. H is

- d-Rev if for any $x \in B_{\mathrm{Ham}}(\mathbf{0}, d) \subseteq \mathbb{B}_2^n$ and $z \in \mathbb{B}_2^n$ where $xH = zH$, then $x = z$.
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Theorem

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Residuated Mappings

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Let $(A, \leq_A)$ and $(B, \leq_B)$ be two partially ordered sets. For mappings $f\colon A \to B$ and $g\colon B \to A$ the pair (f, g) is a *residuated pair*, if there holds

$$f(x) \leq_B y \Longleftrightarrow x \leq_A g(y), \quad \text{for all } x \in A \text{ and } y \in B.$$

We call f residuated if there exists a g such that (f, g) is residuated, and we call g its residual. A residuated pair (f, g) has many nice properties:

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- if A and B are complete lattices then f is residuated if and only if it is a $\vee$-homomorphism and $f(0_A) = 0_B$.

Residuated Mappings on $\mathbb{B}_2^n$

As $\mathbb{B}_2^n$ is a complete lattice with $\vee = +$ and $\wedge = \cdot$, we easily see that

$$f \colon \mathbb{B}_2^n \to \mathbb{B}_2^k \text{ is residuated if and only if } f(x) = xH \text{ for some } H \in \mathbb{B}_2^{n \times k}.$$

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Let $f\colon \mathbb{B}_2^n \to \mathbb{B}_2^k$ be a residuated mapping represented by the $n \times k$ matrix H. Then the residual mapping $g\colon \mathbb{B}_2^k \to \mathbb{B}_2^n$ is given by the assignment $y \mapsto \overline{\overline{y}H^\top}$.

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1. H *is* d*-disjunct.*
2. $(g \circ f)\big|_{B_{\mathrm{Ham}}(\mathbf{0},d)} = \mathrm{id}$.
3. $B_{\mathrm{Ham}}(\mathbf{0},d) \subseteq \mathrm{im}(g)$.
4. $B_{\mathrm{Ham}}(\mathbf{1},d) \subseteq \mathrm{colsp}(H)$.

Group testing schemes

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An (n, k, d)-group testing scheme is a residuated mapping $f\colon \mathbb{B}_2^n \to \mathbb{B}_2^k$ together with a decoder $g\colon \mathbb{B}_2^k \to \mathbb{B}_2^n$ such that $(g \circ f)\big|_{B_{\mathrm{Ham}}(\mathbf{0},d)} = \mathrm{id}$.

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Furthermore, the group testing scheme also allows us to verify if a given test outcome is indeed a valid outcome given the test matrix.

Proposition

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Suggests we should consider error-correcting codes!

Error-Correcting Codes on $\mathbb{B}_2^k$

As a residuated mapping $f\colon \mathbb{B}_2^n \to \mathbb{B}_2^k$ induced by a d-disjunct matrix is injective on $B_{\mathrm{Ham}}(\mathbf{0}, d)$, then for any $\delta \leq d$ we can construct codes $\mathcal{C}_{f,\delta} \subseteq \mathbb{B}_2^k$ as $\mathcal{C}_{f,\delta} = f(B_{\mathrm{Ham}}(\mathbf{0}, \delta))$, with $|\mathcal{C}_{f,\delta}| = \sum_{i=0}^{\delta} \binom{n}{i}$.

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Motivating Example:

Let H be the 7×7 incidence matrix of the Fano plane, which is 2-disjunct. We can define 2 codes:

- $\mathcal{C}_{f,1}$ has minimum distance $3 \Rightarrow$ can identify one infected sample, while also correcting a faulty test.
- $\mathcal{C}_{f,2}$ has minimum distance $2 \Rightarrow$ can identity two infected samples, but cannot correct any faulty tests.

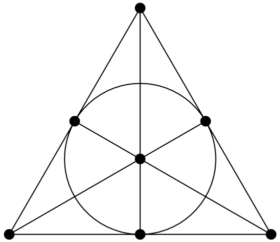

Figure: The Fano plane.

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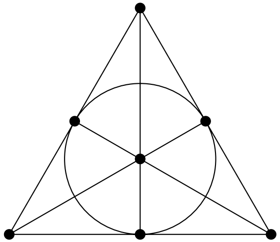

Figure: The Fano plane.

Question: Can we find classes of such codes with efficient decoders?

Inducing codes from Inversive Planes

Definition (Inversive Planes)

An inversive plane of order n is an incidence structure of points and circles, where

1. Any circle has $n+1$ points,
2. A unique circle is incident with any given triple of points,
3. If $p_1 \in c_1$ and $p_2 \notin c_1$, there exists a unique circle c_2 such that $p_1, p_2 \in c_2$,
4. There exists 4 points which are not concircular.

Is also a 3-$(n^2+1, n+1, 1)$-design, or a $S(3, n^2+1, n+1)$ Steiner system.

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Proposition

Let H be the incidence matrix H of an inversive plane of order n and let $f : \mathbb{B}_2^{n(n^2+1)} \to \mathbb{B}_2^{n^2+1}$ be the corresponding residuated mapping. H is then 1-disjunct, and $\mathcal{C}_{f,1} \setminus \{0\}$ is a constant weight code of minimum distance $n+1$.

Thus, inversive planes can be used to generate testing schemes with good error-correcting capabilities, assuming the prevalence of the underlying disease is low.

Partial Linear Spaces

Definition

For $s, t \in \mathbb{N}$, a finite incidence structure (P, L) of points and lines is a *partial linear space* of order (s, t) if

- Every line is incident with $s + 1$ points, and every point is incident with $t + 1$ lines.
- Two different points are connected by at most one line.

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Proof.

Suppose for contradiction that $\ell \neq \ell_i$ for $i = 1, \ldots, m$. Then $|\ell \cap \ell_i| \leq 1$, so

$$s + 1 = |\ell| \leq \left|\ell \cap \bigcup_{i=1}^{m} \ell_i\right| \leq \sum_{i=1}^{m} |\ell \cap \ell_i| \leq m.$$

$\square$

Corollary

An $|L| \times |P|$ incidence matrix H of a partial linear space (P, L) of order (s, t) is s-disjunct.

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Definition (Generalized quadrangle)

A partial linear space (P, L) of order (s, t) is called a generalized quadrangle, $GQ(s, t)$, if for any non-incident point-line pair (p, ℓ), there exists a unique point q on ℓ that is connected with p by a line.

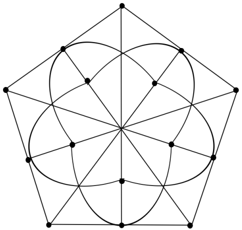

Figure: GQ(2,2), or W(2).

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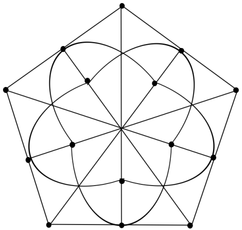

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Corollary

$GQ(s, t)$ yields a $((s + 1)(st + 1), (t + 1)(st + 1), s)$-group testing scheme.

Final remarks

- Group testing is seeing actual proof of concepts, using 1-disjunct 100×20 test matrix (Stoltze et al., 'Combinatorial batching of DNA for ultralow-cost detection of pathogenic variants', Genome Medicine, 2023.), and has received $\geq$€3.000.000 with the intention of clinical implementation in Denmark.

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- Optimal 1-disjunct matrices are solved by Sperner's Theorem, but even for $d = 2$ and $k \in [16]$ the maximal n is not known (A286874, OEIS).
- Try to construct d-disjunct matrices with "nice" properties:
  1. $n \gg k$ and for specific d.
  2. Constant weight columns and rows for samples with "equal dilution".
  3. Sparse matrices.
  4. Induces codes with high minimum distance and an efficient decoder.
     - Assuming different error models; maybe false positives are more likely than false negatives, so channel should not have symmetric cross-over probability.
  5. ...?

Thank you for your attention!