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Regular fat linearized polynomials

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Table of Contents

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Table of Contents

- The *linear set of rank* ρ in $\mathrm{PG}(k-1,q^n)$ associated with an $\mathbb{F}_q$-subspace U of $\mathbb{F}_{q^n}^k$, $\dim_{\mathbb{F}_q} U = \rho$:

$$L_U = \{\langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}} : \mathbf{v} \in U, \mathbf{v} \neq \mathbf{0}\}$$

- $\mathcal{L}_{n,q} = \{\sum_{i=0}^{n-1} a_i X^{q^i} \colon a_0, \ldots, a_{n-1} \in \mathbb{F}_{q^n}\}$. For any $f \in \mathcal{L}_{n,q}$ define $U_f = \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\}$. The *linear set* or rank n associated with f is

$$L_f = L_{U_f} = \{\langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\} \subseteq \mathrm{PG}(1, q^n)$$

- The *weight* w.r.t. L_U of a point $P = \langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}} \in \mathrm{PG}(k-1,q^n)$ is

$$w_{L_U}(P) = w(P) = \dim_{\mathbb{F}_q} \left(\langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}} \cap U \right)$$

- L_U is *scattered* [Blokhuis - Lavrauw 2000] if $\dim_{\mathbb{F}_q} \left(\langle \mathbf{v} \rangle_{\mathbb{F}_{q^n}} \cap U \right) \leq 1$ for all $\mathbf{v} \in \mathbb{F}_{q^n}^k$

Definitions

- $f \in \mathcal{L}_{n,q}$ is *scattered* if L_f is scattered; equivalently,

$$x, y \in \mathbb{F}_{q^n}^*, \quad \frac{f(x)}{x} = \frac{f(y)}{y} \Rightarrow \frac{x}{y} \in \mathbb{F}_q$$

- For $1 < t \mid n$, $L_U \subseteq \mathrm{PG}(k-1, q^n)$ is R- q^t -*partially scattered* if $\dim_{\mathbb{F}_q}\left(\langle \mathbf{v} \rangle_{\mathbb{F}_{q^t}} \cap U\right) \leq 1$ for all $\mathbf{v} \in \mathbb{F}_{q^n}^k$ [Longobardi - Z 2021]

- [Smaldore - Z - Zullo 2024]: Let $n = 2t$, q odd, $\sigma = q^J$, $\gcd(J, t) = 1$, $t \geq 3$, and

$$\phi_{m,\sigma} = X^{\sigma^{t-1}} + X^{\sigma^{2t-1}} + m\left(X^{\sigma} - X^{\sigma^{t+1}}\right) \in \mathcal{L}_{n,\sigma}$$

For any $0 \neq m \in \mathbb{F}_{q^t}$, $\phi_{m,\sigma}$ is R- q^t -partially scattered. If m is neither a $(\sigma - 1)$-th power nor a $(\sigma + 1)$-th power of an element of $E = \{x \in \mathbb{F}_{q^{2t}} : \ \mathrm{Tr}_{q^{2t}/q^t}(x) = 0\}$, the polynomial $\phi_{m,\sigma}$ is scattered.

(r, i)-RFLSs

Main definition

An (r, i)*-regular fat linear set* ((r, i)-RFLS) is one that has precisely r points with weight greater than one, and all of these points have weight i ($r \geq 0$, $i \geq 2$)

(r,i)-RFLSs

Main definition

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- An (r,i)-*regular fat linearized polynomial* ((r,i)-RFLP) is an $f \in \mathcal{L}_{n,q}$ such that L_f is an (r,i)-RFLS
- (r,i)-RFLSs are particular r-fat linear sets, which were defined in [Bartoli - Micheli - Zini - Zullo 2022]
- Any $(0,i)$-RFLS is a scattered linear set, and conversely
- If $f \in \mathcal{L}_{n,q}$ is an (r,i)-regular fat q-polynomial, then

$$|L_f| = q^{n-1} + q^{n-2} + \cdots + q^i + 1 - (r-1)(q^{i-1} + q^{i-2} + \cdots + q)$$

Examples for $r = 1$ or $i = 2$

- The $(1, i)$-RFLS are called *i-clubs* [Fancsali - Sziklai 2006, 2009] and have been widely studied. I'll focus on $r > 1$

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- [De Boeck - Van de Voorde 2022] using [Lavrauw - Van de Voorde 2010] deal with LSs or rank $\rho \leq 4$ in $\mathrm{PG}(1, q^n)$ and rank 5 in $\mathrm{PG}(1, q^5)$. In particular, for $\rho = 4$:
  $|L_U| = q^3 + 1 \ \Rightarrow f$ is either $(1, 3)$-RFLS or $(q + 1, 2)$-RFLS
  $|L_U| = q^3 + q^2 + 1 \ \Rightarrow f$ is $(1, 2)$-RFLS
  $|L_U| = q^3 + q^2 - q + 1 \ \Rightarrow f$ is $(2, 2)$-RFLS
- $(r, 2)$-RFLSs are also investigated in: [Bartoli - Micheli - Zini - Zullo 2022]: $f = X + \delta X^{q^{n-1}}$, $N_{q^n/q}(\delta) = 1$, L_f is $(r, 2)$-RFLS (r is computed)
- Further contributions to $i = 2$: [Csajbók - Marino - Polverino - Z 2018], [Z 2019], [Polverino - Zullo 2020], [Bartoli - Csajbók - Montanucci 2021], [. . .]

Table of Contents

Associated rank-metric code

- *Rank distance* between $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in $\mathbb{F}_{q^n}^m$:

$$d(x, y) = w(x - y) = \dim_{\mathbb{F}_q}(\langle x_1 - y_1, \ldots, x_m - y_m \rangle_{\mathbb{F}_q})$$

- $[m, k, d]_{q^n/q}$*-code*: a k-dimensional $\mathbb{F}_{q^n}$-subspace $\mathcal{C}$ of $\mathbb{F}_{q^n}^m$, where

$$d = \min\{w(x) \colon x \in \mathcal{C},\, x \neq 0\}$$

- Rank-metric *Singleton bound*:

$$nk \leq \max\{m, n\}(\min\{m, n\} - d + 1)$$

- Now I'll disregard the general frame
- Let L_U be an (r, i)-RFLS of rank ρ in $\mathrm{PG}(k-1, q^n)$. Take $G \in \mathbb{F}_{q^n}^{k \times (nk - \rho)}$ having as columns an $\mathbb{F}_q$-basis of

$$U^{\perp'} = \{x \in \mathbb{F}_{q^n}^k \colon \mathrm{Tr}_{q^n/q}(x \cdot u) = 0,\, \forall u \in U\}$$

- Define $\mathcal{C} \leq \mathbb{F}_{q^n}^{nk - \rho}$ as the rowspace of G

Associated rank-metric code

Proposition

If $i < n$, the rank-metric code $\mathcal{C}$ associated with L_U, an (r, i)-RFLS of rank ρ in $\mathrm{PG}(k-1, q^n)$, is an $[nk - \rho, k, n - i]_{q^n/q}$-code with

- $r(q^n - 1)$ codewords of weight $n - i$
- $(|L_U| - r)(q^n - 1)$ codewords of weight $n - 1$
- $(q^{nk} - 1) - |L_U|(q^n - 1)$ codewords of weight n, and
- $$|L_U| = \frac{q^\rho - 1 - r(q^i - q)}{q - 1}$$

Associated rank-metric code

- A direct application of the Singleton bound gives, for an (r, i)-RFLS of rank ρ in $\mathrm{PG}(k-1, q^n)$

$$\rho \leq nki/(i+1) \tag{1}$$

- From the MacWilliams identities

$$r \geq \frac{(q^{2\rho-nk}-1)\binom{n}{2}_q}{(q^n-1)\binom{i}{2}_q} \tag{2}$$

- (1), (2) are useless in $\mathrm{PG}(1, q^n)$

Table of Contents

Points of complementary weights

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A *linear set with complementary weights* has two points such that the sum of the weights of the points equals the rank of the linear set [Napolitano - Polverino - Santonastaso - Zullo 2022]

Theorem [Napolitano - Polverino - Santonastaso - Zullo 2022]

Let L_W be an $\mathbb{F}_q$-linear set of rank $\rho \leq n$ in $\mathrm{PG}(1, q^n)$ for which there exist two distinct points $P, Q \in L_W$ such that $w(P) = s$, $w(Q) = s'$ and $s + s' = \rho$. Then, L_W is $\mathrm{PGL}(2, q^n)$-equivalent to a linear set L_U where $U = S \times S'$, for some $\mathbb{F}_q$-subspaces S and S' of $\mathbb{F}_{q^n}$ with $\dim_q(S) = s$, $\dim_q(S') = s'$. Also, $S \cap S' = \{0\}$ can be assumed.

Points of complementary weights

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Theorem [Napolitano - Polverino - Santonastaso - Zullo 2022]

Let L_W be an $\mathbb{F}_q$-linear set of rank n in $\mathrm{PG}(1,q^n)$ for which there exist two distinct points $P, Q \in L_W$ such that $w(P) = s$, $w(Q) = s'$ and $s + s' = n$. Then, for some $\mathbb{F}_q$-subspaces S and S' of $\mathbb{F}_{q^n}$ with $\dim_q(S) = s$, $\dim_q(S') = s'$, $\mathbb{F}_{q^n} = S \oplus S'$, up to projectivities

$$L_W = L_{p_{S,S'}} = \{\langle (x, p_{S,S'}(x))\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$$

where $p_{S,S'}$ is the projection map related to the direct sum $S \oplus S'$.

The polynomial representation of the projection is

$$p_{S,S'} = \sum_{j=0}^{n-1} \left(\sum_{i=t}^{n-1} \xi_i {\xi_i^*}^{q^j} \right) X^{q^j}$$

where $\{\xi_i\}$ and $\{\xi_j^*\}$ are dual $\mathbb{F}_q$-bases of $\mathbb{F}_{q^n}$ related to S and S'
We have a lack of neat polynomial representations

Points of complementary weights

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Theorem [Napolitano - Polverino - Santonastaso - Zullo 2022]

Let $1 < t < n$ and $n = \ell t$. There exist $\mathbb{F}_q$-linear sets of rank ρ in $\mathrm{PG}(1, q^n)$ with one point of weight t, one point of weight s and all others of weight one for the following values of n, k and s:

- n even, $\rho = t + s$ and any $s \in \{1, \dots, n/2\}$;
- n odd, $\rho = t + s$ and any $s \in \{1, \dots, \frac{n-t}{2}\}$.

Corollary

If t divides n, there is a $(2, t)$-RFLS in $\mathrm{PG}(1, q^n)$.

Points of complementary weights

Polynomials $\phi_{m,\sigma}$ give a simple polynomial representation for some $(2,t)$-RFLS in [Napolitano - Polverino - Santonastaso - Zullo 2022]:

Theorem [Smaldore - Z - Zullo 202x]

1. Let $t \geq 3$. For any m in the form $m = w^{\sigma+1} \neq 0$, $w \in E = \{x \in \mathbb{F}_{q^{2t}} \colon \mathrm{Tr}_{q^{2t}/q^t}(x) = 0\}$, the linear set associated with $\phi_{m,\sigma} = X^{\sigma^{t-1}} + X^{\sigma^{2t-1}} + m\left(X^{\sigma} - X^{\sigma^{t+1}}\right)$ is projectively equivalent to $L_{T\times T'}$ where $T, T' = \{wx \pm x^{\sigma^{t-1}} \mid x \in \mathbb{F}_{q^t}\}$.
2. For t odd, $\langle(1,0)\rangle_{\mathbb{F}_{q^n}}$ and $\langle(0,1)\rangle_{\mathbb{F}_{q^n}}$ are the only points of weight t of $L_{T\times T'}$, and $L_{T\times T'}$ is equivalent to a $(2,t)$-RFLS in [Napolitano - Polverino - Santonastaso - Zullo 2022].

- [Zullo 2023] (k,i)-RFLS with $i \leq n/2$ in $\mathrm{PG}(k-1,q^n)$
- Further constructions of $(2,i)$-RFLSs in [Alfarano - Jurrius - Neri - Zullo 202x]

☞ To our knowledge, there are no examples of (r, i)-RFLSs in $\mathrm{PG}(1, q^n)$ with $r > 2$ and $i > 2$ in the literature.

In other words, apart from the examples I am going to show, we don't know of any linear sets with exactly r points of weight i and the rest with weight one for $r > 2$ and $i > 2$. If you know of any more, please tell us!

Table of Contents

Theorem [Smaldore - Z - Zullo 202x]

Let q be odd, $t \geq 3$, $\gcd(s,t) = 1$, $w \in E = \{x \in \mathbb{F}_{q^{2t}} : \mathrm{Tr}_{q^{2t}/q^t}(x) = 0\}$, $w \neq 0$, $\mathrm{N}_{q^t/q}(w^2) \neq (-1)^t$, $I \leq_{\mathbb{F}_q} \mathbb{F}_{q^t}$, $\dim_{\mathbb{F}_q} I = i > 1$. Define

$$T = T_{s,w,I} = \{x + wx^{q^s} : x \in I\} \subseteq \mathbb{F}_{q^{2t}}$$

Then for any $k > 1$, L_{T^k} is a $((q^k-1)/(q-1), i)$-RFLS of rank ki in $\mathrm{PG}(k-1, q^{2t})$. The points of weight i are precisely the elements of $\mathrm{PG}(k-1,q)$, i.e. $\langle (a_1, \ldots, a_k) \rangle_{\mathbb{F}_{q^{2t}}}$ with $(0, \ldots, 0) \neq (a_1, \ldots, a_k) \in \mathbb{F}_q^k$.

In particular we have a $(q+1, i)$-RFLS in $\mathrm{PG}(1, q^{2t})$ for any $i = 2, \ldots, t$

Remark

For $q > 3$ there exists $w \in E$, $w \neq 0$ such that $\mathrm{N}_{q^t/q}(w^2) \neq -1$, while $\mathrm{N}_{q^t/q}(w^2) \neq 1$ holds for any $w \in E$.

Polynomial form

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Theorem [Smaldore - Z - Zullo 202x]

For any $\mu \in \mathbb{F}_{q^t}$ such that $\mathrm{N}_{q^t/q}(\mu) = 1$, $\mu \neq 1$, any rank n $L_{T^2} \subseteq \mathrm{PG}(1, q^{2t})$ is equivalent up to the action of $\Gamma\mathrm{L}(2, q^{2t})$ to L_f, where

$$f = (\mu^{q^s} - 1)\left((\mu+1)X^{q^t} - 2w^{-q^{t-s}}(X^{q^{t-s}} - X^{q^{2t-s}})\right) + (\mu - 1)\left((\mu^{q^s}+1)X^{q^t} + 2w\mu^{q^s}(X^{q^s} + X^{q^{t+s}})\right)$$

or, for t even and taking $\mu = -1$, to $L_{\phi_{m,\sigma}}$, where $\phi_{m,\sigma} = X^{\sigma^{t-1}} + X^{\sigma^{2t-1}} + m\left(X^{\sigma} - X^{\sigma^{t+1}}\right)$ is the polynomial introduced in [Smaldore - Z - Zullo 2024], m being a nonzero $(\sigma+1)$-power of an element of E and $\sigma = q^{t-s}$.

Theorem [Smaldore - Z - Zullo 202x]

All L_{T^k} are R- q^t -partially scattered linear sets.

Table of Contents

$\phi_{m,\sigma}$ is always RFLS

Theorem [Smaldore - Z - Zullo 202x]

Let $\phi_{m,\sigma} = X^{\sigma^{t-1}} + X^{\sigma^{2t-1}} + m\left(X^{\sigma} - X^{\sigma^{t+1}}\right) \in \mathbb{F}_{q^{2t}}[X]$, $t \geq 3$, q odd, $m \in \mathbb{F}^*_{q^t}$.

- If m is a $(\sigma - 1)$-power of an element of $E = \{x \in \mathbb{F}_{q^n} : x^{q^t} + x = 0\}$, then $L_{\phi_{m,\sigma}}$ is an $(r, 2)$-RFLS.
- If m is a $(\sigma + 1)$-power of an element of E and t is odd, then $L_{\phi_{m,\sigma}}$ is a $(2, t)$-RFLS.
- If m is a $(\sigma + 1)$-power of an element of E and t is even, then $L_{\phi_{m,\sigma}}$ is a $(q + 1, t)$-RFLS.
- Otherwise $L_{\phi_{m,\sigma}}$ is a $(0, -)$-RFLS, i.e., scattered.

More examples

$(q + 1, t)$-RFLSs of rank $2t$ in $\mathrm{PG}(1, q^{\ell t})$, $\ell > 2$, $\ell \mid q^t - 1 \ldots$

Have a nice day!