Representability of uniform q-matroids

Ferdinando Zullo
Università degli Studi della Campania "Luigi Vanvitelli"

(Joint work with G.N. Alfarano, M. Borello, R. Jurrius, A. Neri and O. Polverino)

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Matroids vs q-Matroids

Matroid

$\mathscr{M}$ **is a pair** $([n], r)$ **where** $r\colon 2^{[n]} \to \mathbb{Z}$
$\forall A, B \subseteq [n]$
(r1) (Boundness) $0 \leq r(A) \leq |A|$
(r2) (Monotonicity)
$A \subseteq B \Rightarrow r(A) \leq r(B)$
(r3) (Submodularity)
$r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

Rank of $\mathscr{M}$: $r([n])$

q-Matroid

$\mathscr{M}$ **is a pair** $(\mathbb{F}_q^n, \rho)$ **where**
$\rho\colon \mathscr{L}(\mathbb{F}_q^n) \to \mathbb{Z}$ $\forall A, B \in \mathscr{L}(\mathbb{F}_q^n)$
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Rank of $\mathscr{M}$: $\rho(\mathbb{F}_q^n)$

Jurrius, R. P. M. J., and G. R. Pellikaan. "Defining the q-analogue of a matroid." The Electronic Journal of Combinatorics 25.3

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Alfarano, Byrne, Ceria, Gluesing-Luerssen, Jany, Jurrius and Pellikaan...

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Example: Uniform matroid $U_{k,n}$

$U_{k,n} = ([n], r)$, where
$r(A) = \min\{|A|, k\}, \ \forall A \subseteq [n]$

Example: Uniform q-matroid $\mathscr{U}_{k,n}(q)$

$\mathscr{U}_{k,n}(q) = (\mathbb{F}_q^n, \rho)$, where
$\rho(A) = \min\{\dim(A), k\}, \ \forall A \in \mathscr{L}(\mathbb{F}_q^n)$

Example: Matroids from codes

$G = (G^1 \cdots G^n) \in \mathbb{F}_q^{k\times n}$

$\mathscr{M}[G] = ([n], r)$ **where**

$r(A) = \mathrm{rk}(G^s : s \in A), \ \forall A \subseteq [n]$

$\mathscr{M}[G]$ **is a representable matroid**

Hamming metric codes

Example: q-Matroids from codes

$G \in \mathbb{F}_{q^m}^{k\times n}$

$\mathscr{M}[G] = (\mathscr{L}(\mathbb{F}_q^n), \rho)$ **where**

$\forall A \in \mathscr{L}(\mathbb{F}_q^n)$ **consider** $\mathrm{colspan}(Y_A) = A$

$\rho(A) = \mathrm{rk}(GY_A)$

$\mathscr{M}[G]$ **is an $\mathbb{F}_{q^m}$-representable q-matroid**

Rank metric codes

$U_{k,n}$ is representable if q is large enough

$$G = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \cdots & \alpha_n^{k-1} \end{pmatrix}$$

$\mathcal{U}_{k,n}(q)$ is $\mathbb{F}_{q^m}$- representable $\Leftrightarrow m \geq n$

$$G = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^q & \alpha_2^q & \cdots & \alpha_n^q \\ \vdots & & & \vdots \\ \alpha_1^{q^{k-1}} & \alpha_2^{q^{k-1}} & \cdots & \alpha_n^{q^{k-1}} \end{pmatrix}$$

$\Downarrow$

Existence of MDS codes/arcs

$\Downarrow$

Existence of MRD codes/ scattered subspaces w.r.t. hyperplanes

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There are matroids that are not representable

Vamos, Peter. "On the representation of independence structures." Unpublished manuscript 120 (1968).

There are q-matroids that are not representable

* Byrne, E., Ceria, M., Ionica, S., & Jurrius, R. (2024). Weighted subspace designs from q-polymatroids. Journal of Combinatorial Theory, Series A, 201, 105799.
* Gluesing-Luerssen, H., & Jany, B. (2022). q-Polymatroids and their relation to rank-metric codes. Journal of Algebraic Combinatorics, 56(3), 725-753.

Direct sum of uniform q-matroids

* $\mathcal{M}_i = (\mathbb{F}_q^{n_i}, \rho_i),\ i = 1,2$
* $\mathbb{F}_q^n = \mathbb{F}_q^{n_1} \oplus \mathbb{F}_q^{n_2}$
* $\pi_i \colon \mathbb{F}_q^n \to \mathbb{F}_q^{n_i}$, **projections onto the first n_1 and last n_2 coordinates**

Ceria, M., & Jurrius, R. (2024). The direct sum of q-matroids. Journal of Algebraic Combinatorics, 59(2), 291-330.

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Direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ is

$(\mathbb{F}_q^n, \rho)$ **where**

$$\rho(V) = \dim(V) + \min_{X \leq V}(\rho_1(\pi_1(X)) + \rho_2(\pi_2(X)) - \dim(X))$$

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Gluesing-Luerssen, H., & Jany, B. (2024). Decompositions of-matroids using cyclic flats. SIAM Journal on Discrete Mathematics, 38(4), 2940-2970.

Associativity → **Direct sum t q-matroids**

If $\mathscr{M}_1$ and $\mathscr{M}_2$ are two q-matroids $\mathbb{F}_{q^m}$-representable $\not\Longrightarrow$ $\mathscr{M}_1 \oplus \mathscr{M}_2$ is $\mathbb{F}_{q^m}$-representable

$$\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$$

$$G = \begin{pmatrix} 1 & \omega & 0 & \omega + 1 \\ 0 & 0 & 1 & \omega \end{pmatrix} \in \mathbb{F}_4^{2\times 4}$$

$\mathscr{M}_G \oplus \mathscr{M}_G$ is not $\mathbb{F}_{2^m}$-representable for every m

Gluesing-Luerssen, H., & Jany, B. (2025). Representability of the direct sum of q-matroids. Journal of Algebraic Combinatorics, 61(4), 51.

If $\mathscr{M}_1$ and $\mathscr{M}_2$ are two q-matroids $\mathbb{F}_{q^m}$-representable

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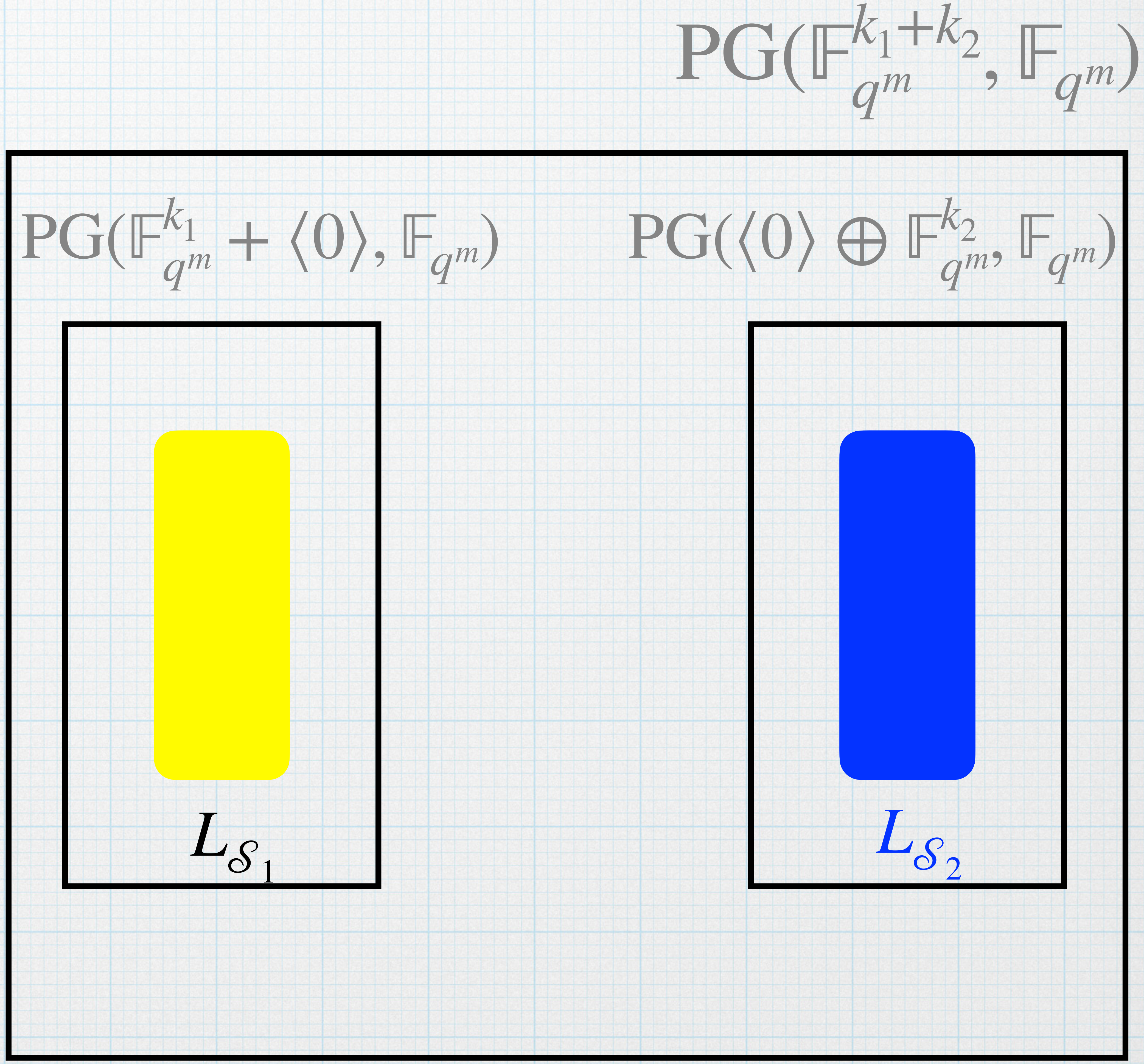
$\mathscr{M}_1 \oplus \mathscr{M}_2$ is $\mathbb{F}_{q^m}$-representable

$\mathscr{M}_1 = \mathscr{U}_{k_1,n_1}(q)$ and $\mathscr{M}_2 = \mathscr{U}_{k_2,n_2}(q)$

Alfarano, G. N., Jurrius, R., Neri, A., & FZ. Representability of the direct sum of uniform q-matroids. Accepted for publication in Combinatorial Theory.

If $\mathscr{M}_1$ and $\mathscr{M}_2$ are two q-matroids $\mathbb{F}_{q^m}$-representable $\overset{?}{\Longrightarrow}$ $\mathscr{M}_1 \oplus \mathscr{M}_2$ is $\mathbb{F}_{q^m}$-representable



Theorem (Alfarano, Jurrius, Neri and FZ)

$\mathscr{U}_{k_1,n_1}(q) \oplus \mathscr{U}_{k_2,n_2}(q)$ is $\mathbb{F}_{q^m}$-representable $\Leftrightarrow$

There exist $\mathcal{S}_1 \leq_{\mathbb{F}_q} \mathbb{F}_{q^m}^{k_1}$ and $\mathcal{S}_2 \leq_{\mathbb{F}_q} \mathbb{F}_{q^m}^{k_2}$ of dimension n_1 and n_2 (resp.)

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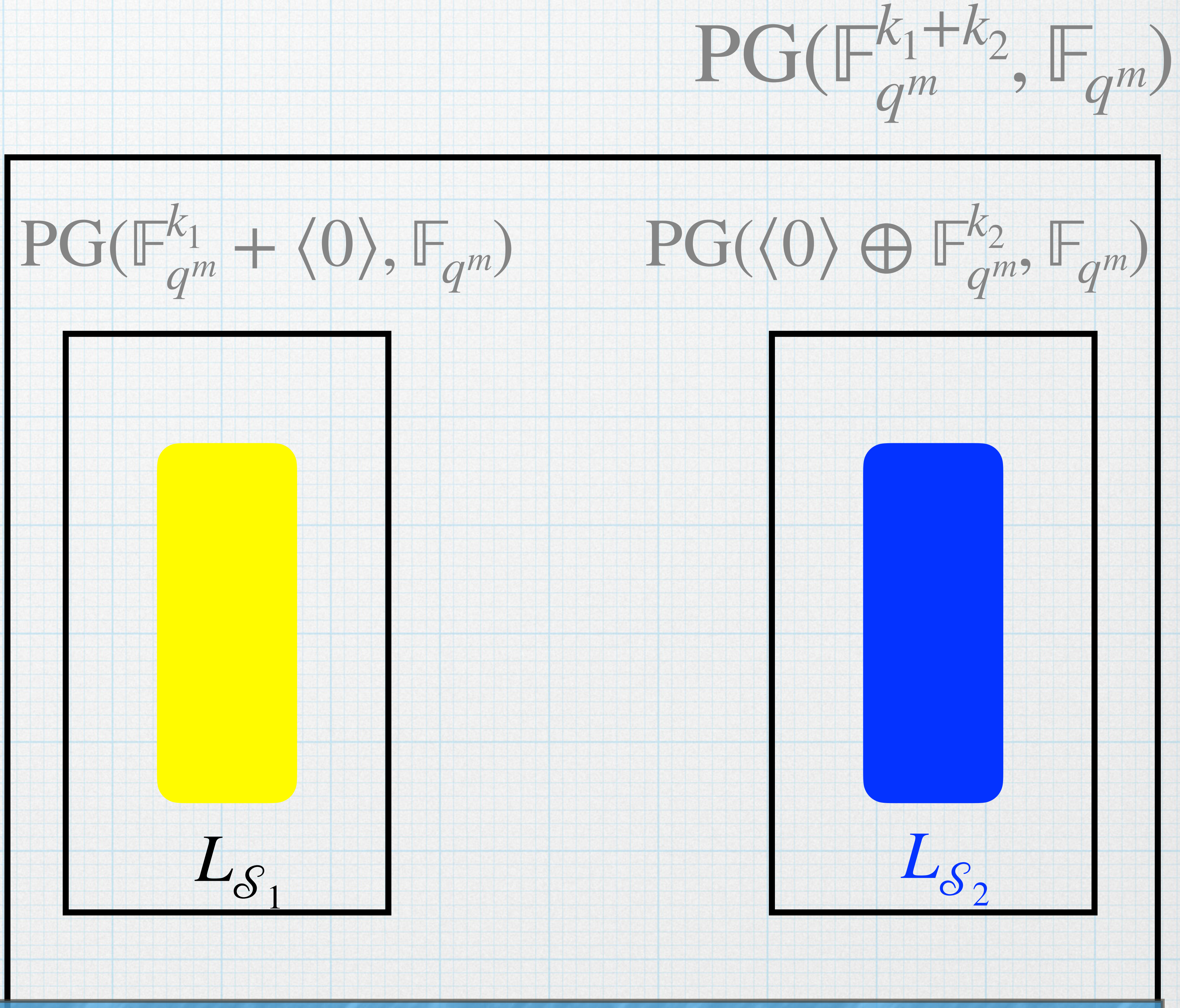

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Scattered w.r.t. the hyperplanes

Lunardon, G. (2017). MRD-codes and linear sets. Journal of Combinatorial Theory, Series A, 149, 1-20.

Sheekey, J., & Van de Voorde, G. (2020). Rank-metric codes, linear sets, and their duality. Designs, Codes and Cryptography, 88(4), 655-675.

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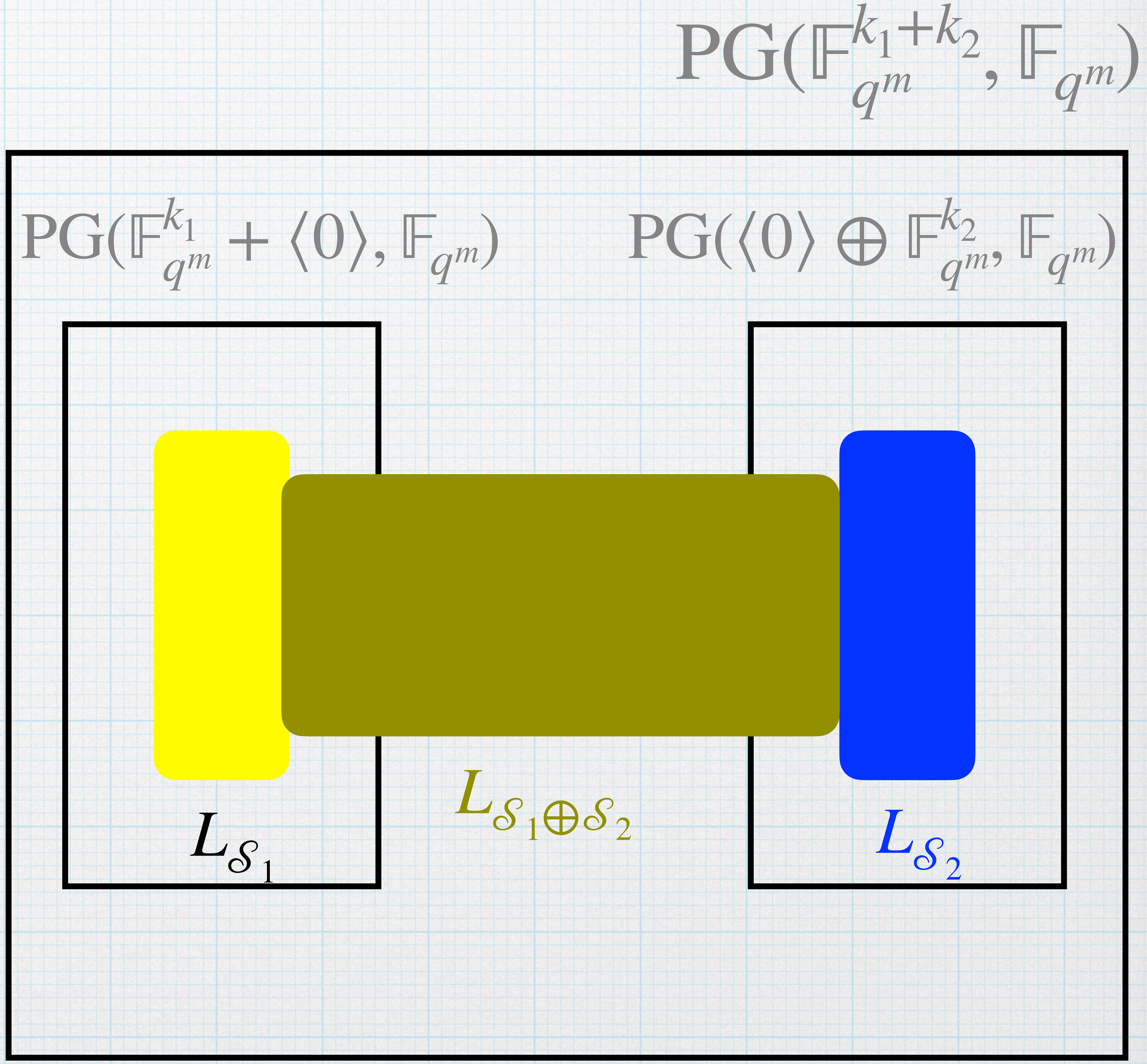

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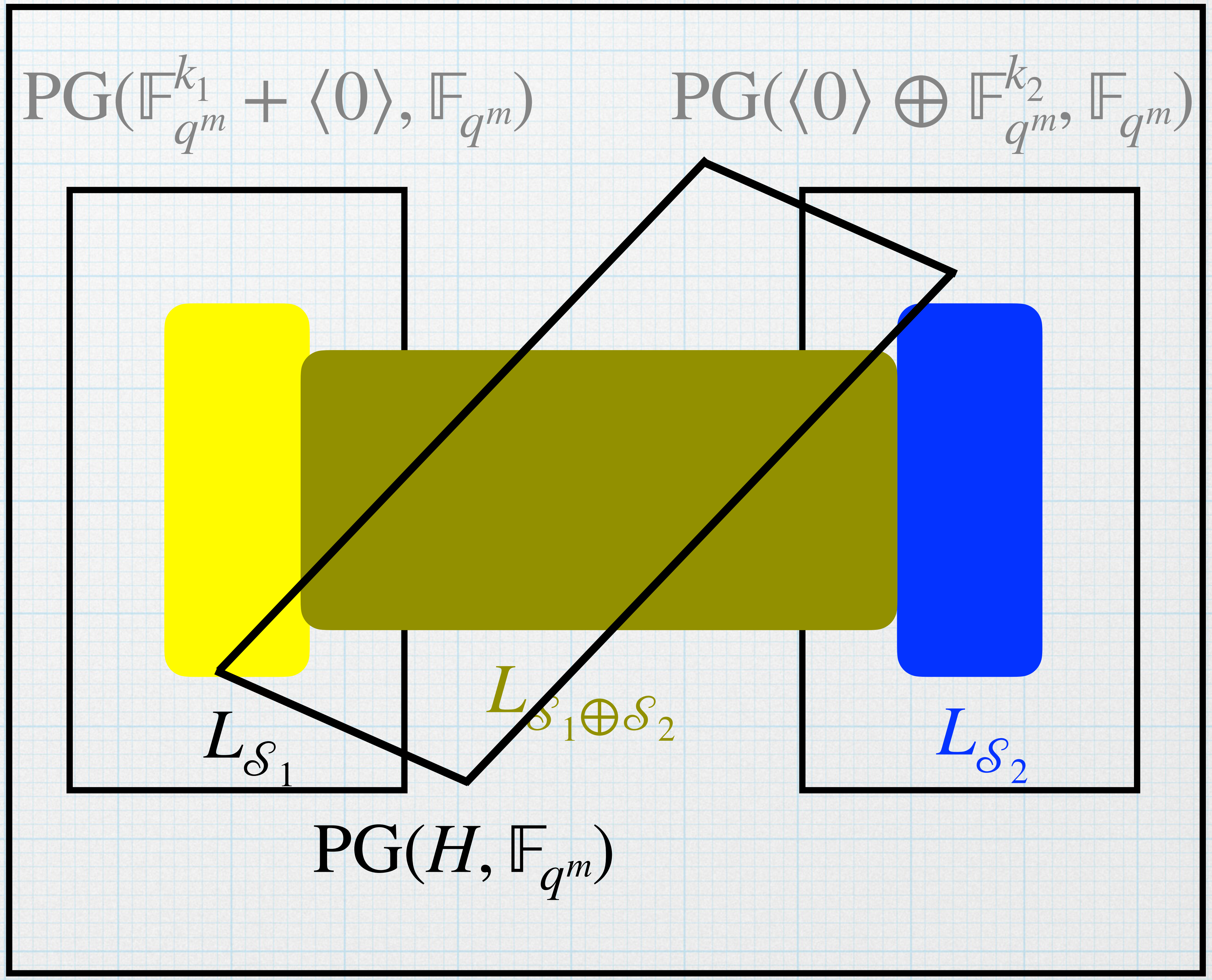

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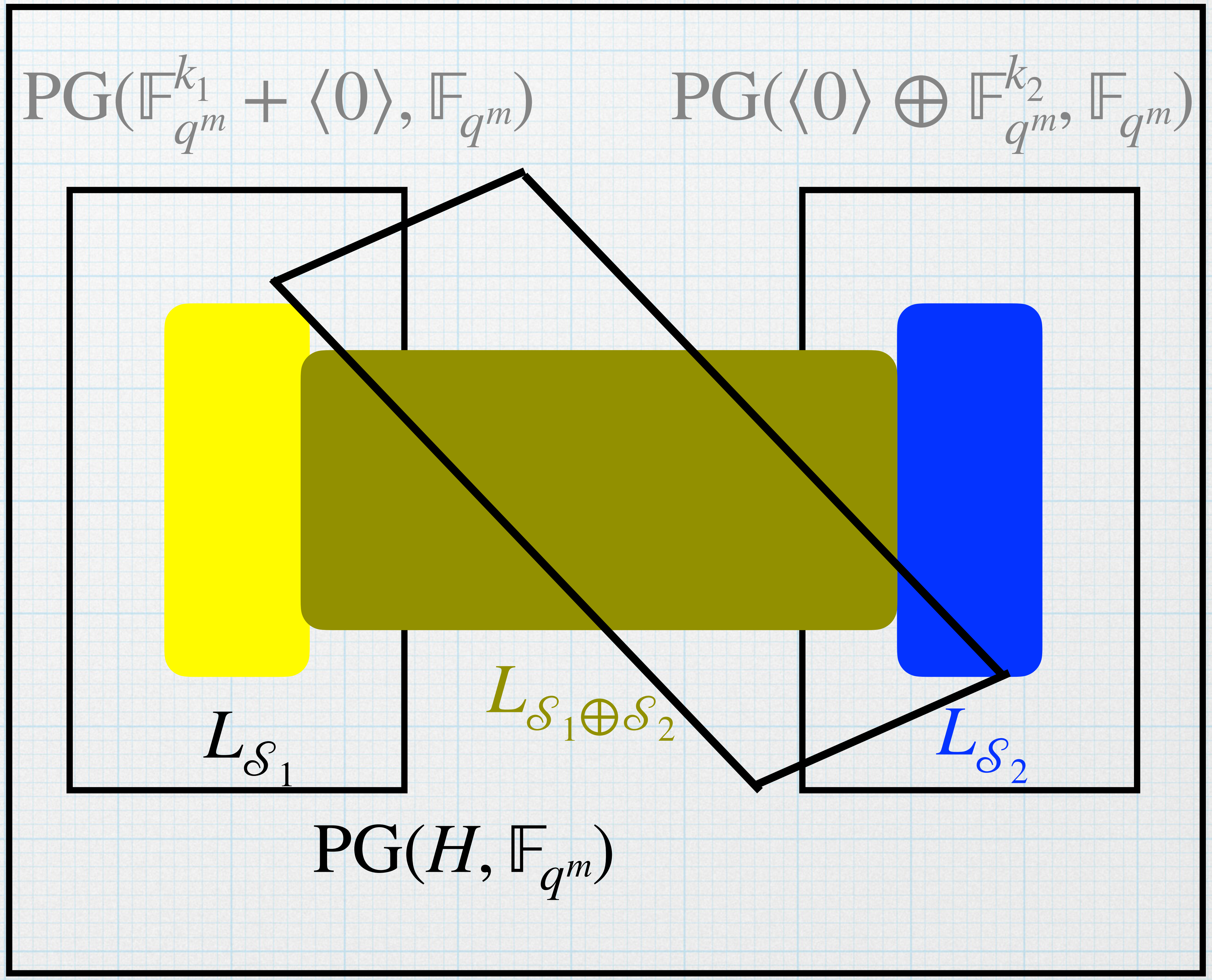
$\mathrm{PG}(\mathbb{F}_{q^m}^{k_1+k_2}, \mathbb{F}_{q^m})$



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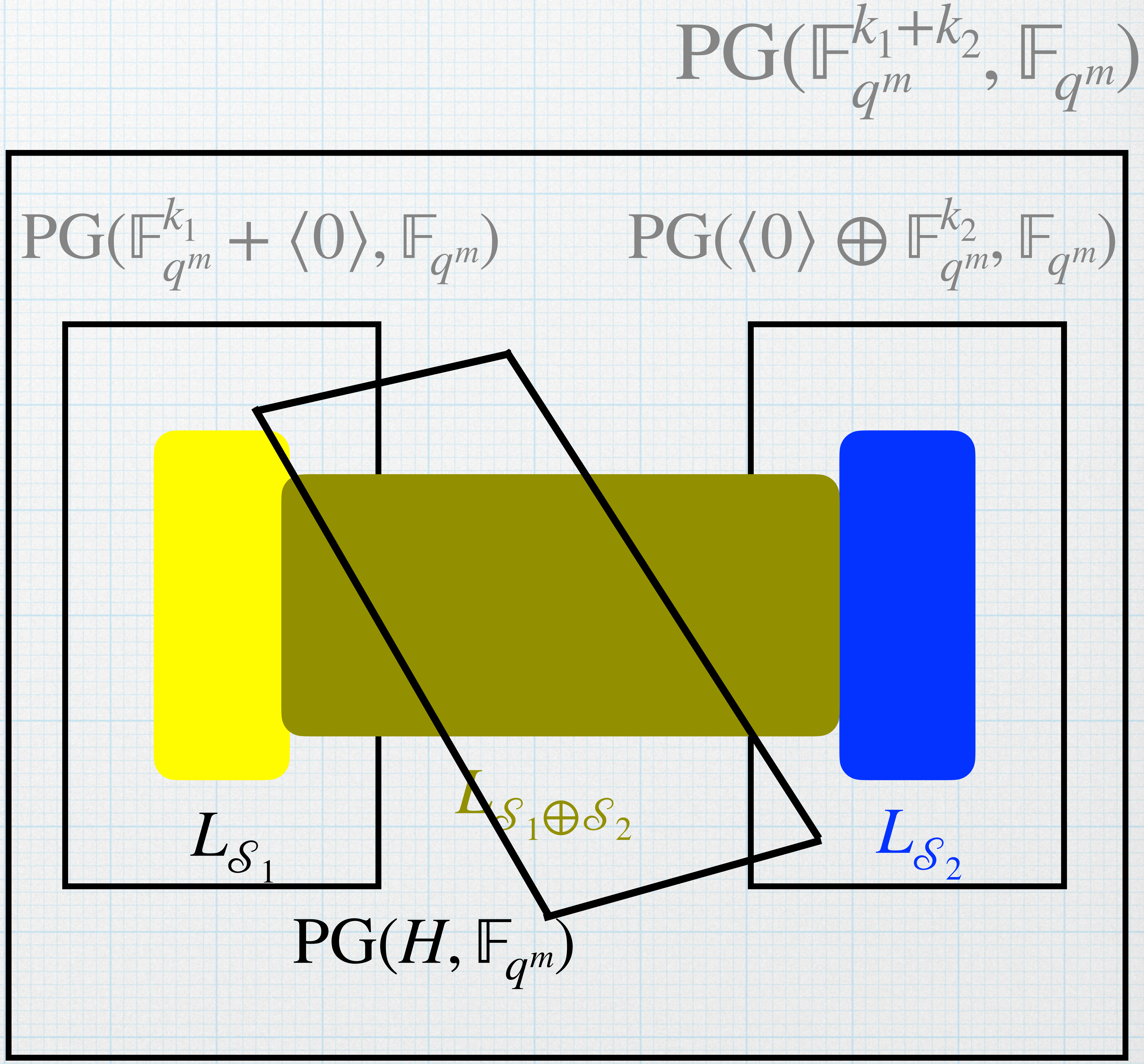

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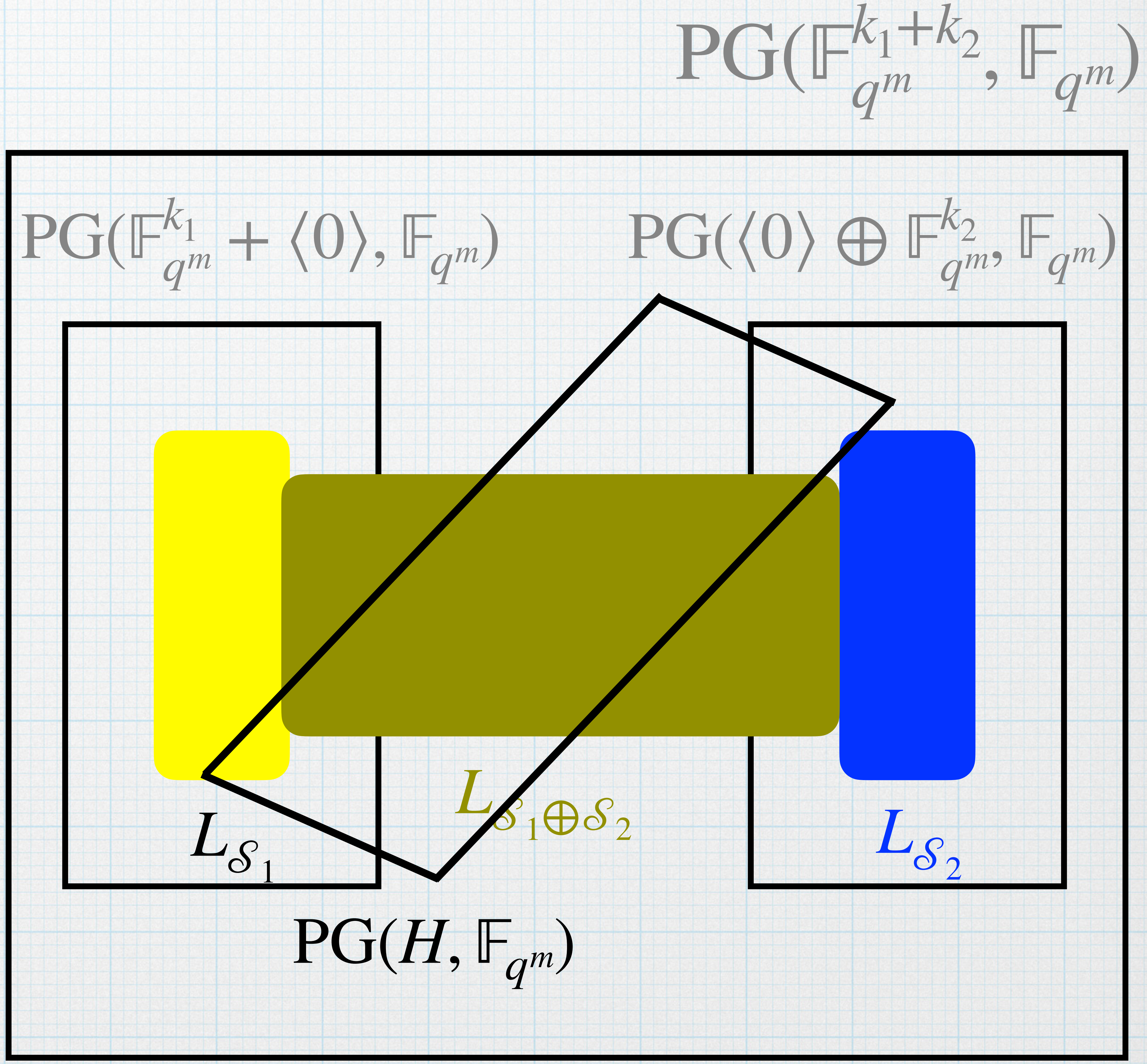
$\mathscr{S}_1 \oplus \mathscr{S}_2$ **is $(\Lambda_{k_1+k_2-1,k_1,k_2}, k_1 + k_2 - 1)$-evasive**



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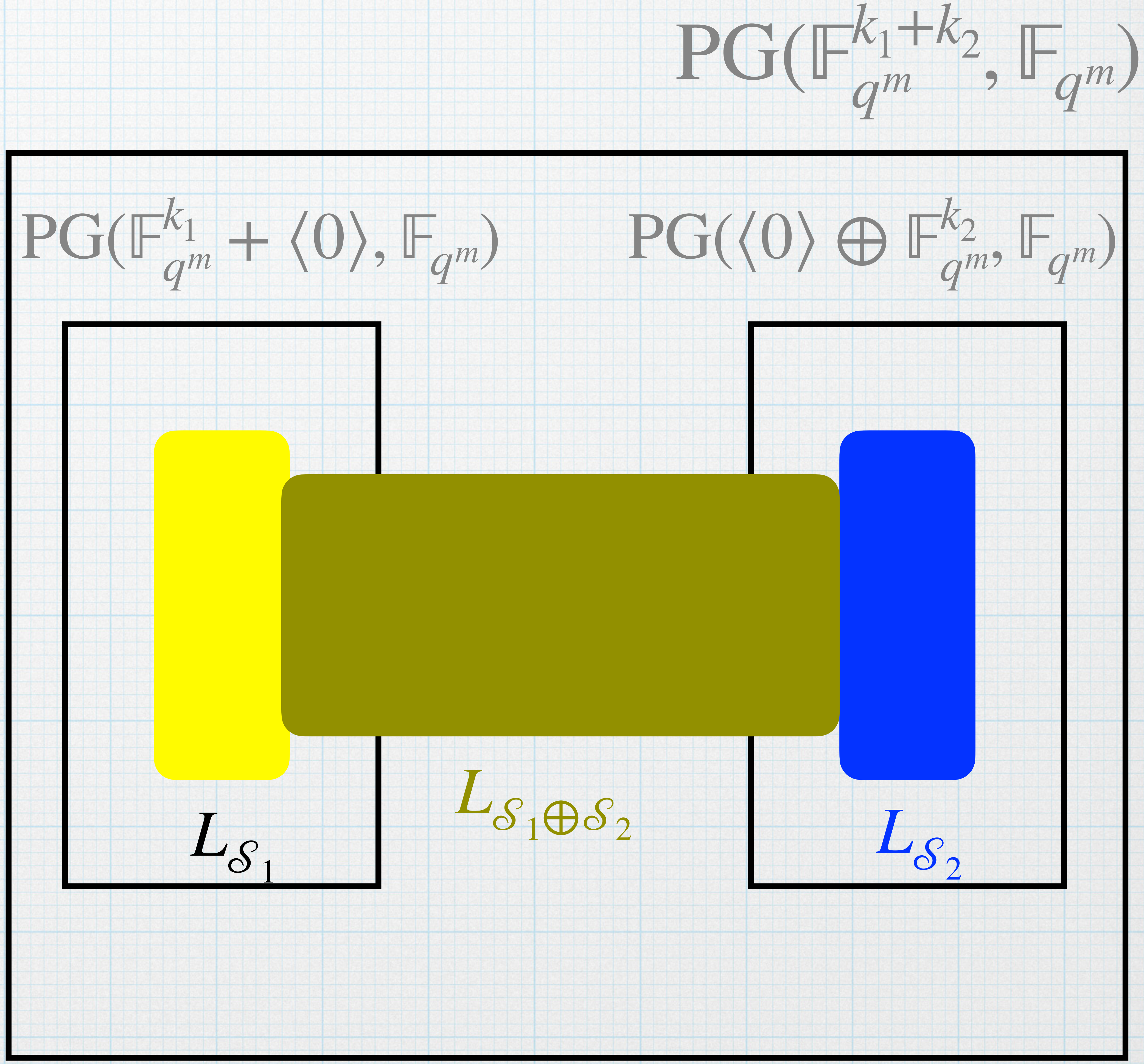

Theorem (Alfarano, Jurrius, Neri and FZ)

$\mathscr{U}_{k_1,n_1}(q) \oplus \mathscr{U}_{k_2,n_2}(q)$ is $\mathbb{F}_{q^m}$-representable for $m = m_1 m_2$ with $n_i \leq m_i$ and $\gcd(m_1, m_2) = 1$

Construction

$\mathscr{S}_1 = \{(x, x^q, \ldots, x^{q^{k_1-1}}) : x \in \mathbb{F}_{q^{n_1}}\}$

$\mathscr{S}_2 = \{(y, y^q, \ldots, y^{q^{k_2-1}}) : y \in \mathbb{F}_{q^{n_2}}\}$

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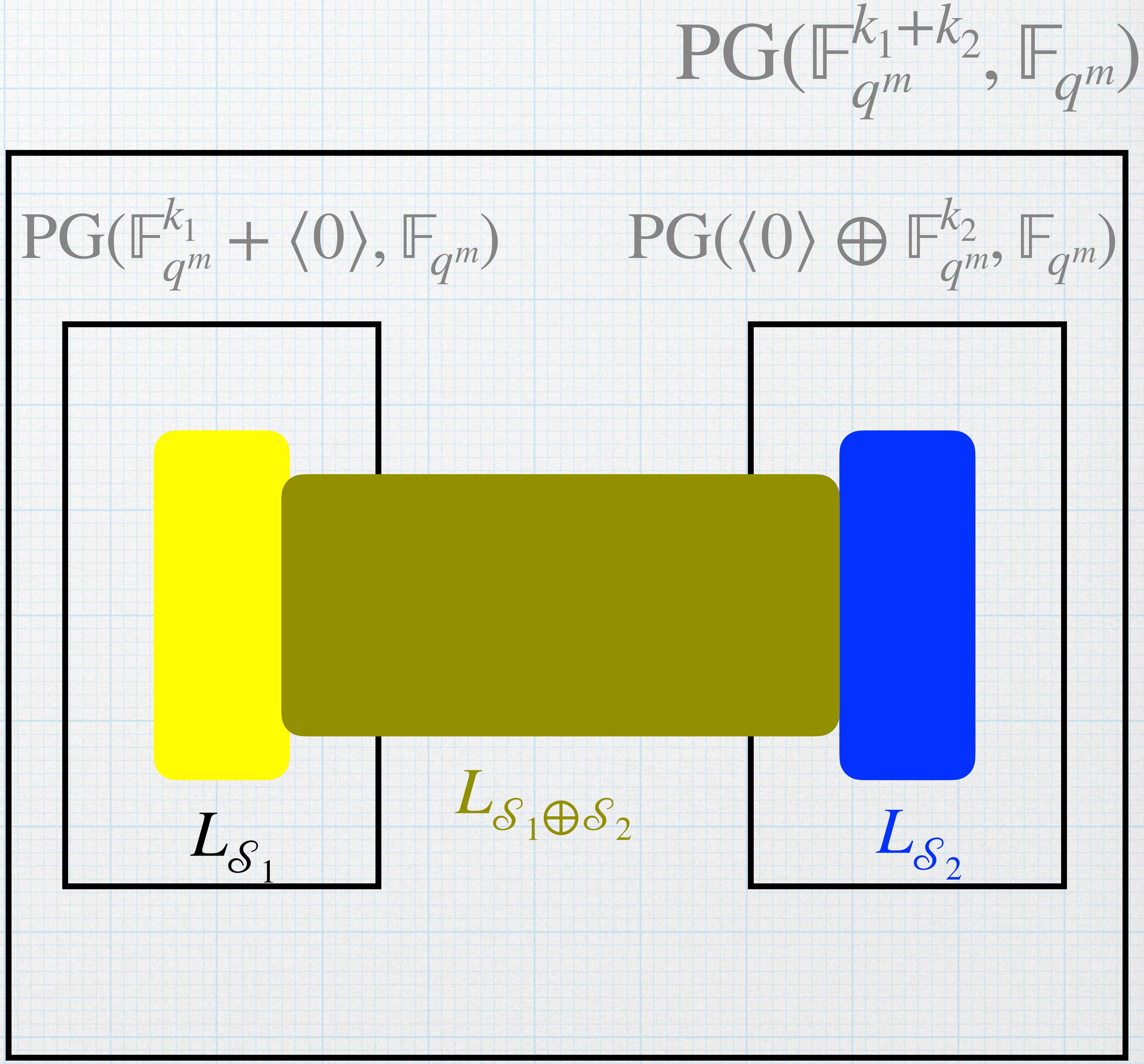

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$\mathscr{S}_1 \oplus \mathscr{S}_2$ is $(\Lambda_{k_1+k_2-1,k_1,k_2}, k_1 + k_2 - 1)$-evasive

Extension to $\mathscr{U}_{k_1,n_1}(q) \oplus \ldots \oplus \mathscr{U}_{k_t,n_t}(q)$

Characterization of m for which $\mathscr{M}$ is $\mathbb{F}_{q^m}$-representable ?

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$$\mathscr{U}_{1,n_1}(q) \oplus \mathscr{U}_{1,n_2}(q)$$

Existence of $S, T \leq_{\mathbb{F}_q} \mathbb{F}_{q^m}$ **s.t.** $L_{S\times T} = \{\langle (s,t)\rangle_{\mathbb{F}_{q^m}} : s \in S, t \in T\}$ **and**
$w_{L_{S\times T}}(\langle (s,t)\rangle_{\mathbb{F}_{q^m}}) = \dim_{\mathbb{F}_q}((S\times T)\cap \langle (s,t)\rangle_{\mathbb{F}_{q^m}}) \leq 1$
if $st \neq 0$

Napolitano, V., Polverino, O., Santonastaso, P., & Zullo, F. (2022). Linear sets on the projective line with complementary weights. Discrete Mathematics, 345(7), 112890.

Corrado's talk!

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m **is even and** $m \geq 2\max\{n_1, n_2\}$

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Corrado's talk!

Other conditions

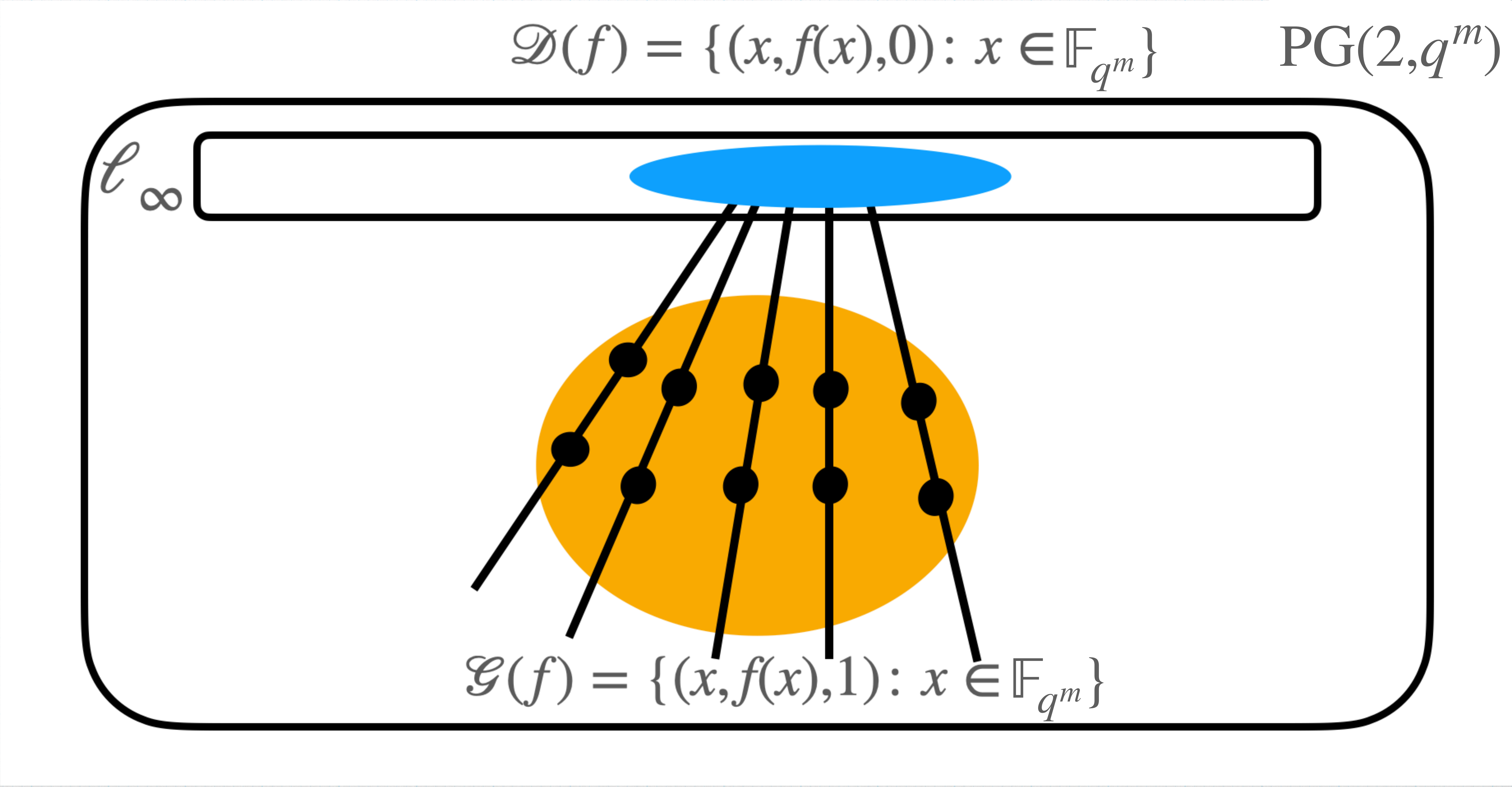
Neri, A., & Stanojkovski, M. (2024). A proof of the Etzion-Silberstein conjecture for monotone and MDS-constructible Ferrers diagrams. Journal of Combinatorial Theory, Series A, 208, 105937.

Characterization of m for which $\mathscr{M}$ is $\mathbb{F}_{q^m}$-representable ?

$\mathcal{U}_{1,n_1}(q) \oplus \mathcal{U}_{1,n_2}(q)$

$$f(x) = \sum_{i=0}^{m-1} a_i x^{q^i} \in \mathbb{F}_{q^m}[x]$$

$\mathcal{D}(f) = \{(x, f(x), 0) : x \in \mathbb{F}_{q^m}\}$



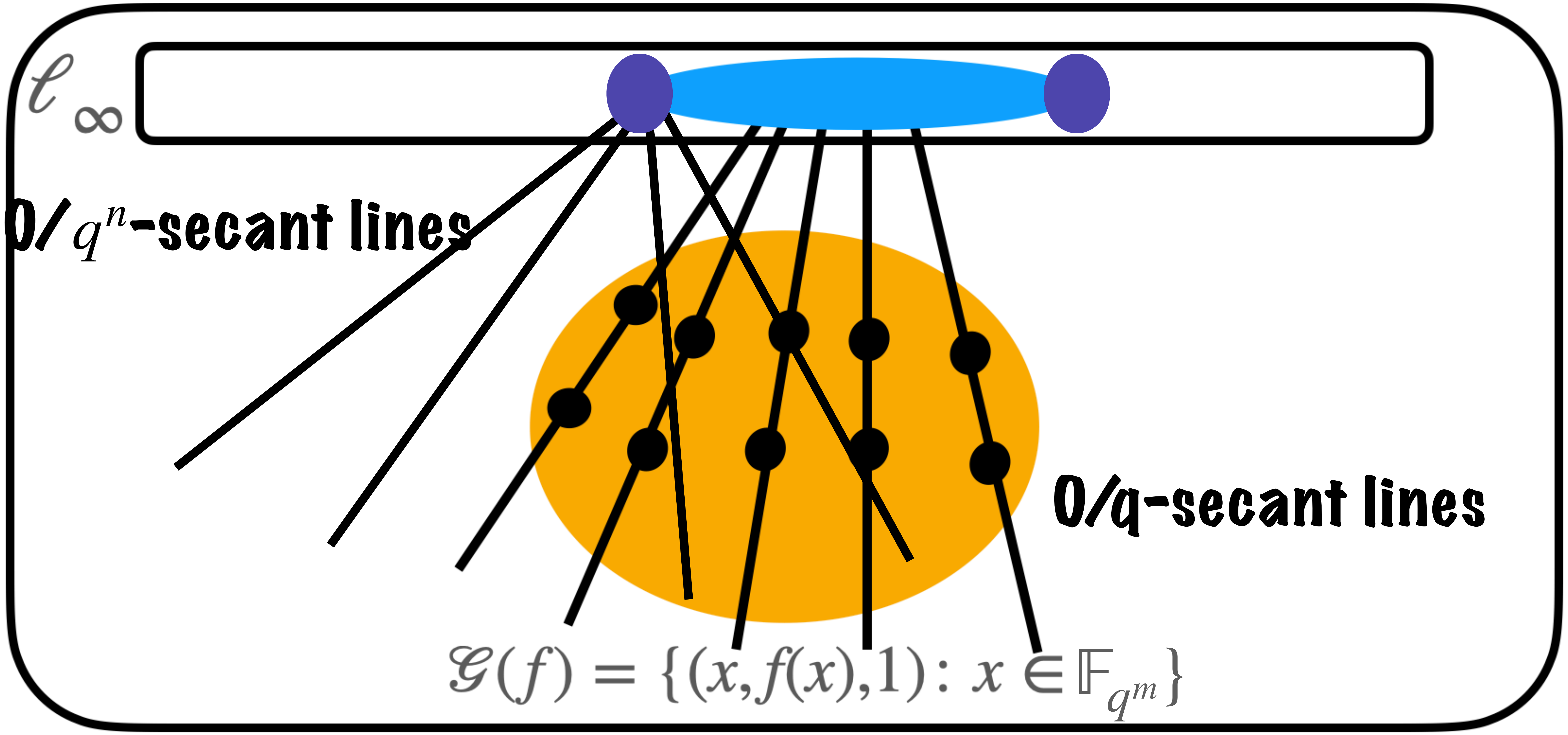
Characterization of m for which $\mathscr{M}$ is $\mathbb{F}_{q^m}$-representable ?

$$\mathscr{U}_{1,n_1}(q) \oplus \mathscr{U}_{1,n_2}(q)$$

$$f(x) = x^{q^{n-1}} + x^{q^{2n-1}} + \mu(x^q - x^{q^{n+1}}) \in \mathbb{F}_{q^{2n}}[x]$$

Smaldore V., Zanella C. And FZ. Regular fat linearised polynomials.
Ongoing project.

$$\mathscr{D}(f) = \{(x, f(x), 0) : x \in \mathbb{F}_{q^m}\} \qquad \mathrm{PG}(2, q^m)$$



$$\mathscr{G}(f) = \{(x, f(x), 1) : x \in \mathbb{F}_{q^m}\}$$

Characterization of m for which $\mathcal{M}$ is $\mathbb{F}_{q^m}$-representable ?

$\mathcal{U}_{1,n_1}(q) \oplus \mathcal{U}_{1,n_2}(q)$

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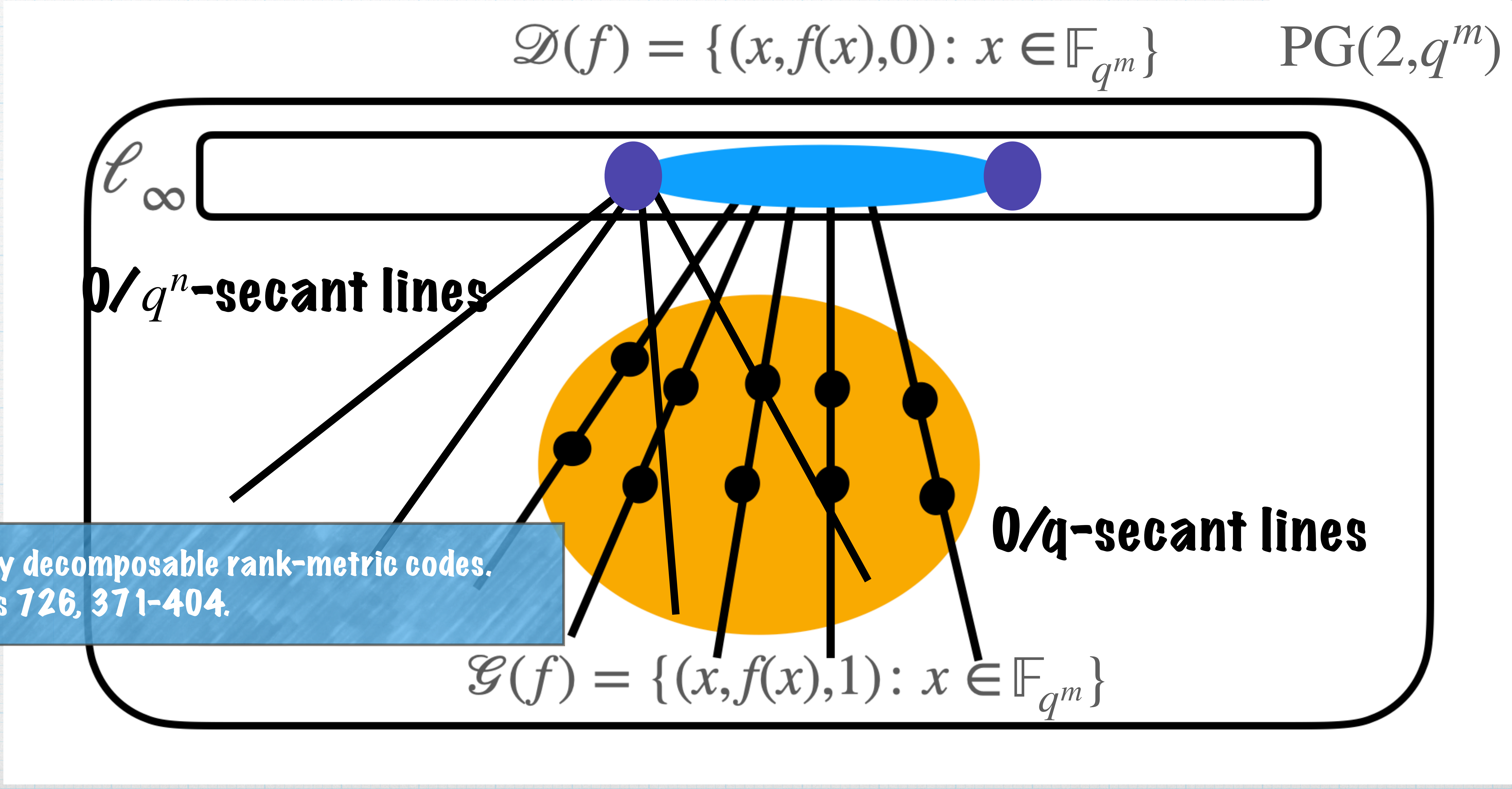

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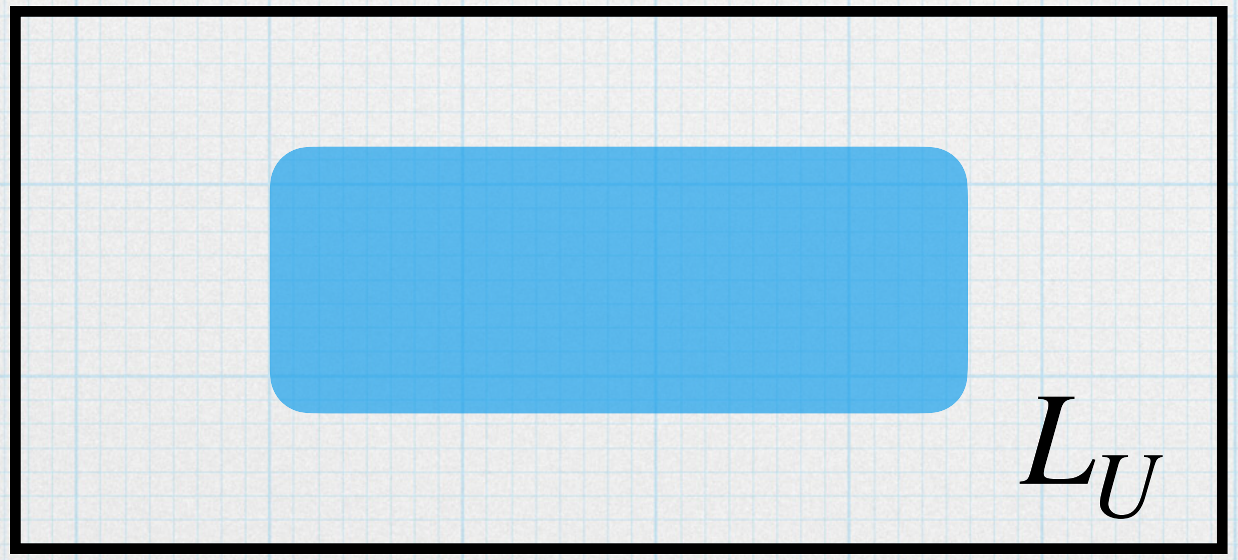
Delsarte duality and direct sum of uniform q-matroids

Delsarte duality

L_U is an $\mathbb{F}_q$-linear set of rank n in $\mathrm{PG}(k-1, q^m)$



$\Lambda = \mathrm{PG}(k-1, q^m)$

Lunardon, G., & Polverino, O. (2004). Translation ovoids in orthogonal polar spaces. Forum Mathematicum (Vol. 16, pp. 663-669).

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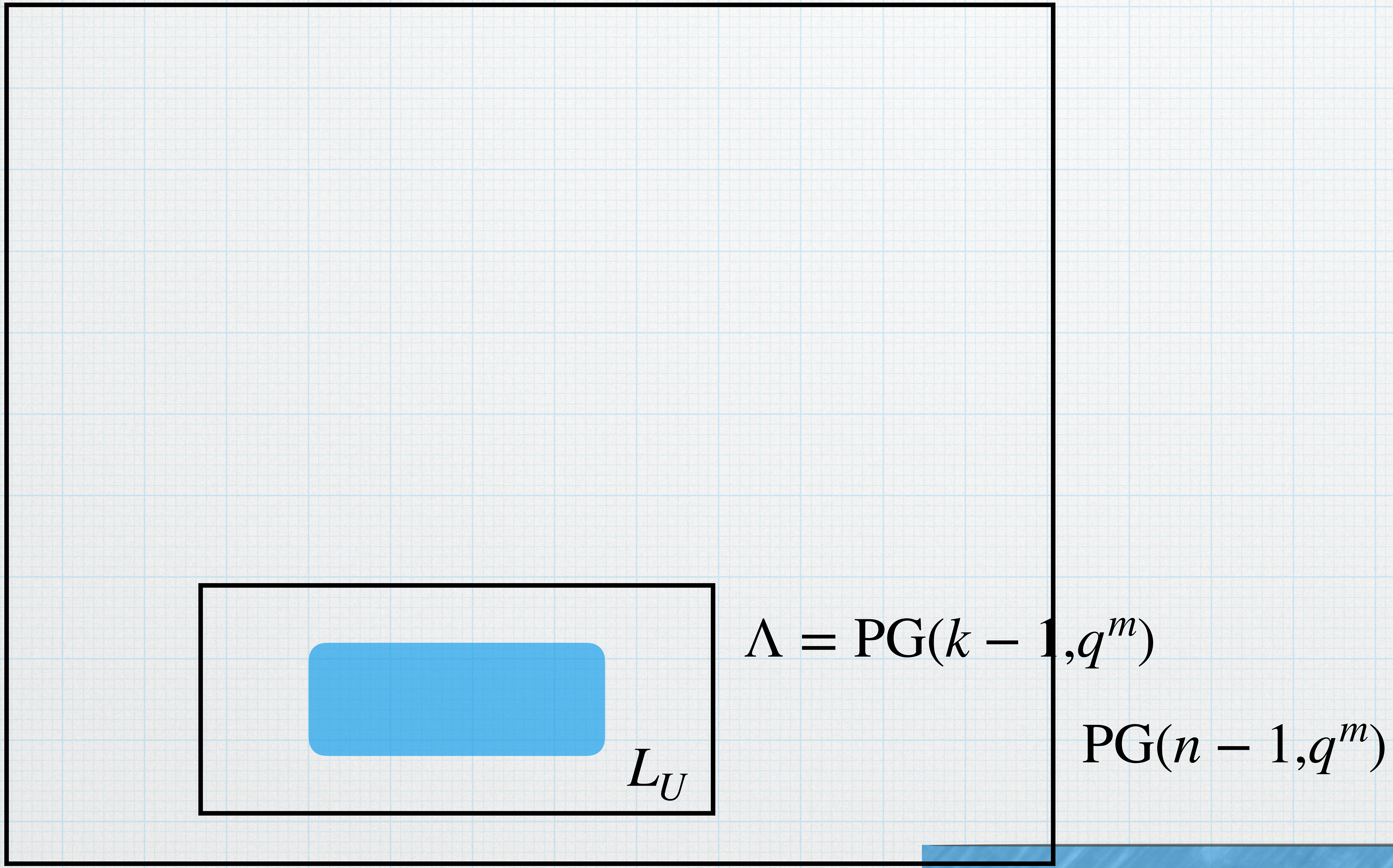

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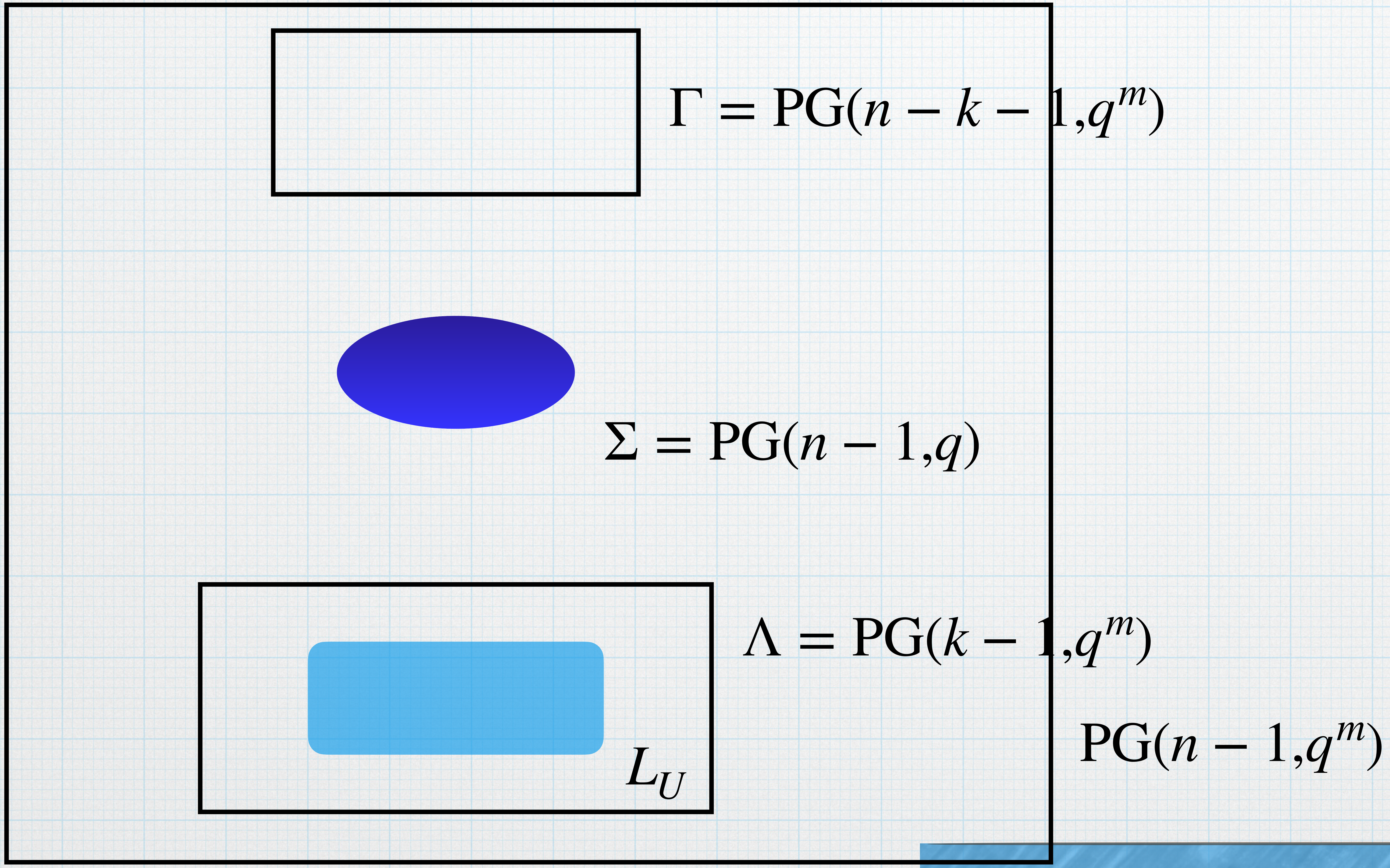

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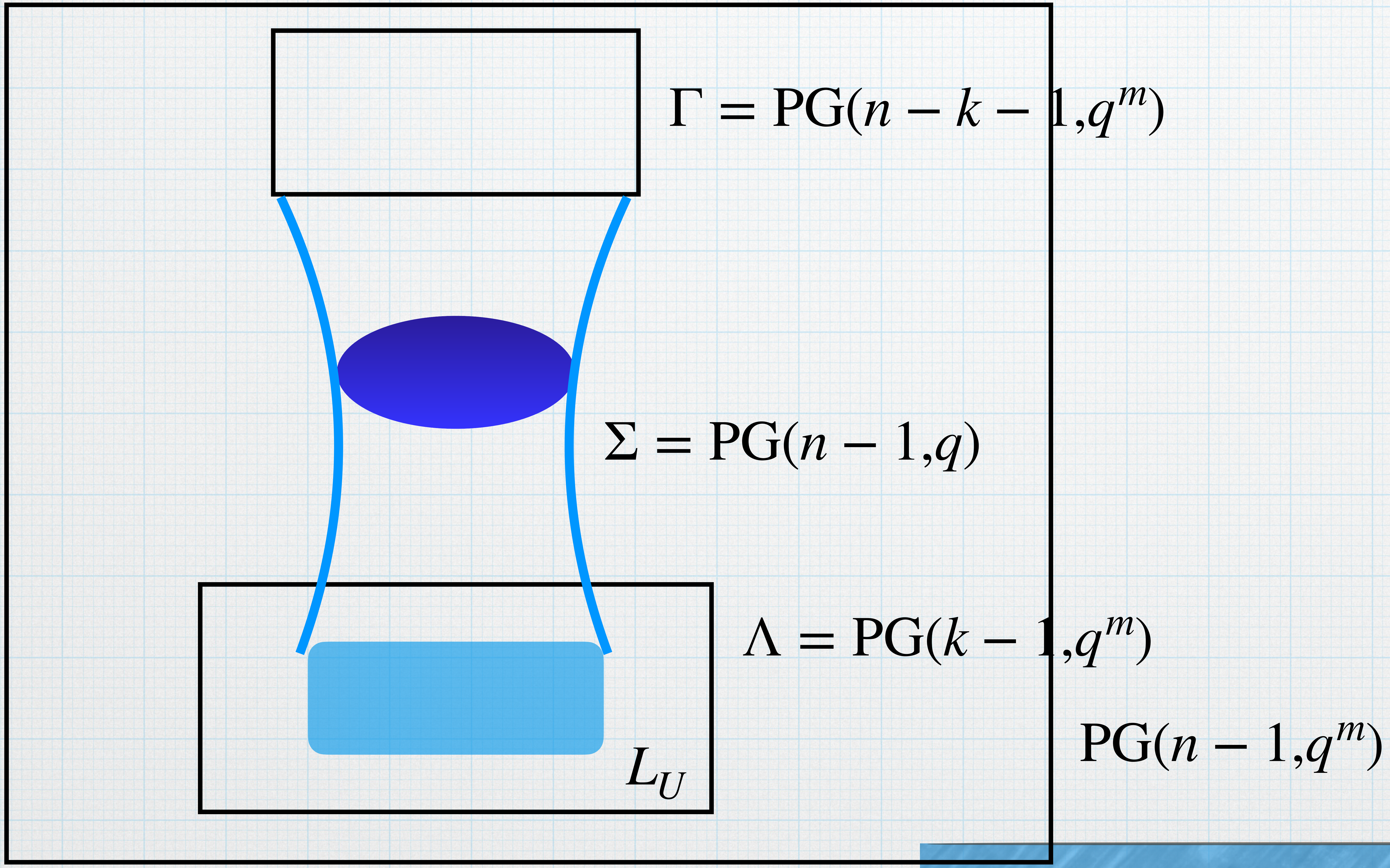

$$L_U = \mathrm{proj}_{\Gamma,\Lambda}(\Sigma)$$

Lunardon, G., & Polverino, O. (2004). Translation ovoids in orthogonal polar spaces. Forum Mathematicum (Vol. 16, pp. 663-669).

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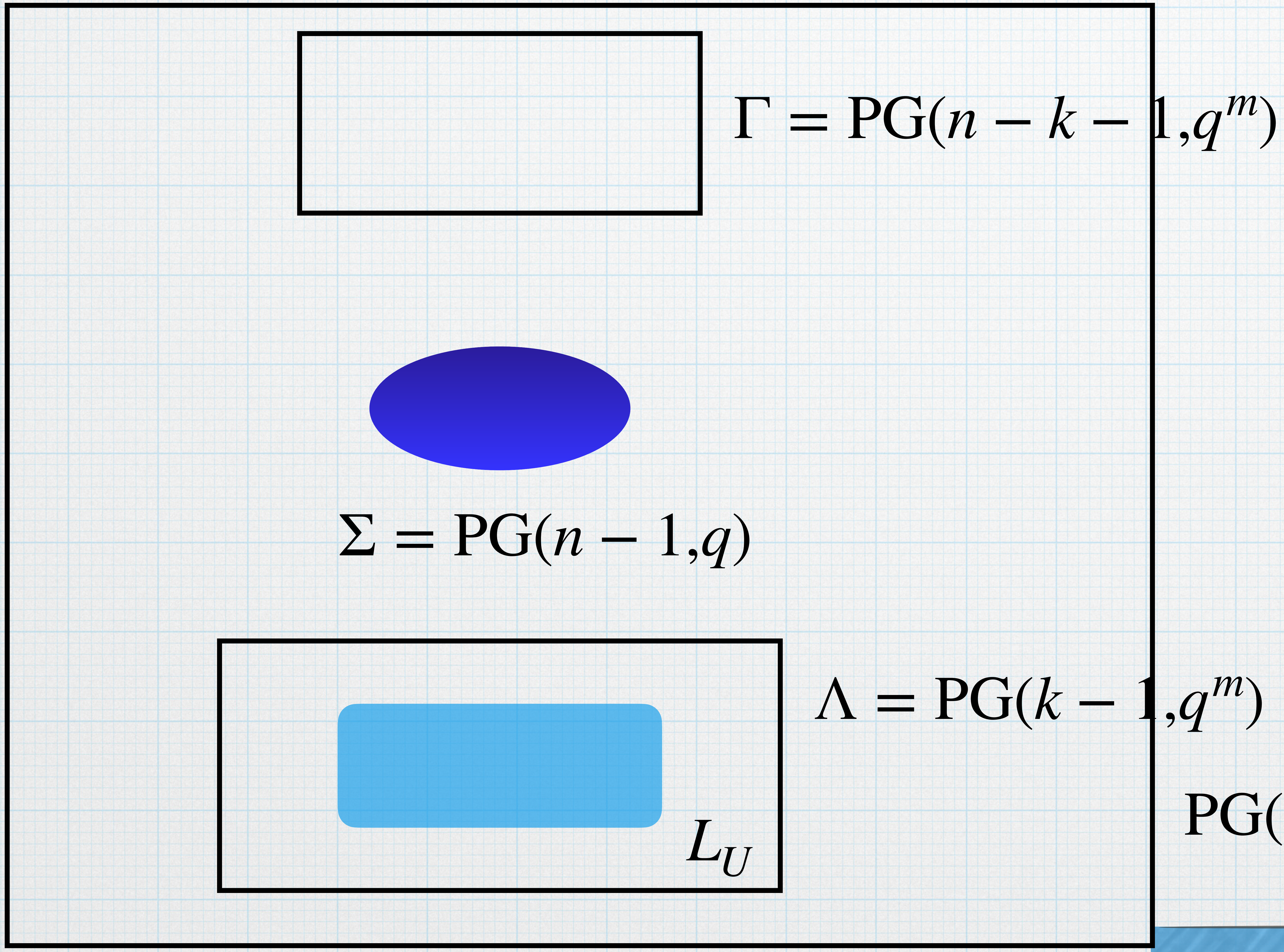

$\mathrm{PG}(n-1,q^m)$

$\tau' \rightarrow$ polarity of $\mathrm{PG}(n-1,q)$

$(\perp')$

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$(\perp)$

Csajbók, B., Marino, G., Polverino, O., & FZ (2021). Generalising the scattered property of subspaces. Combinatorica, 41(2), 237-262.

Delsarte duality

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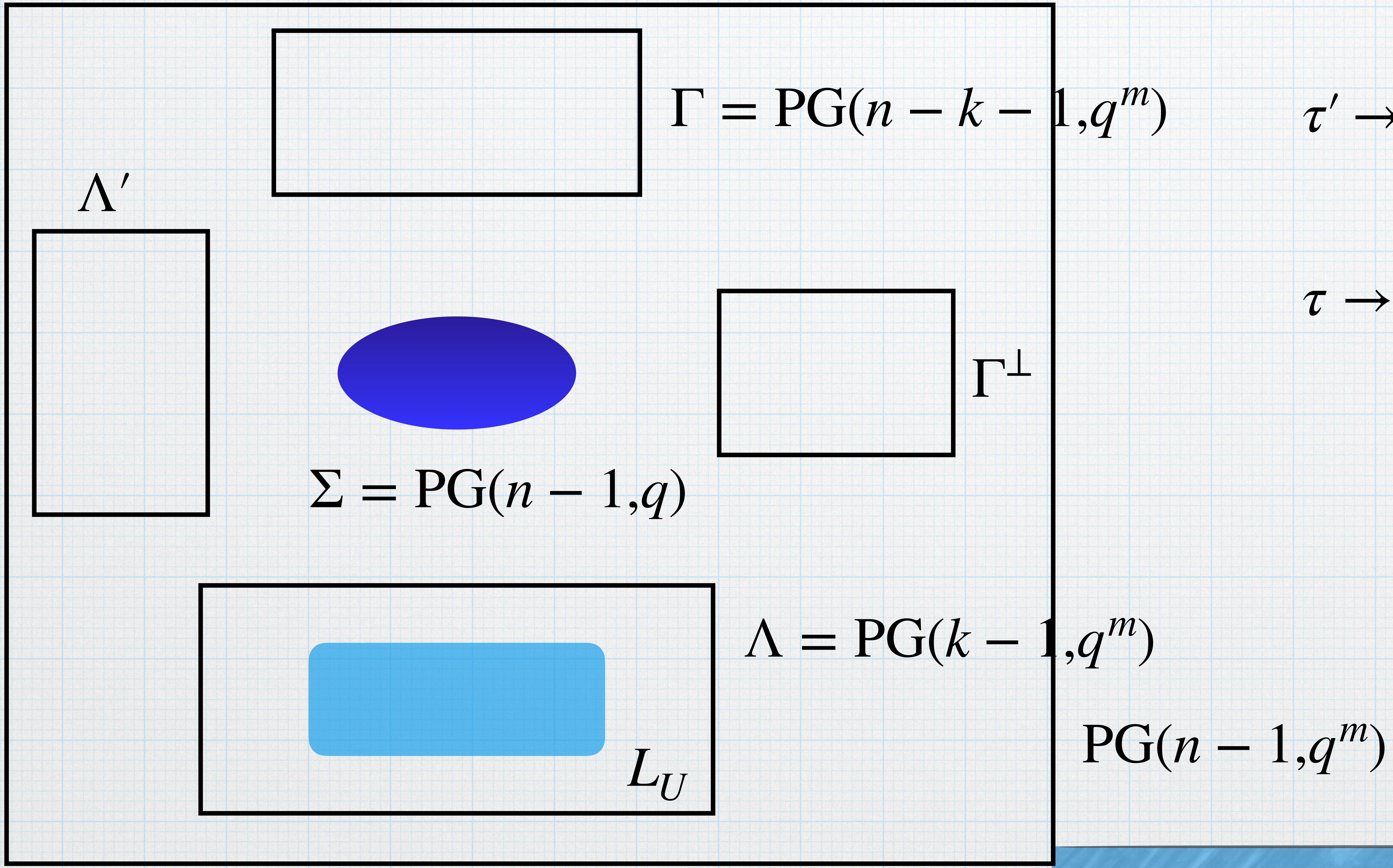

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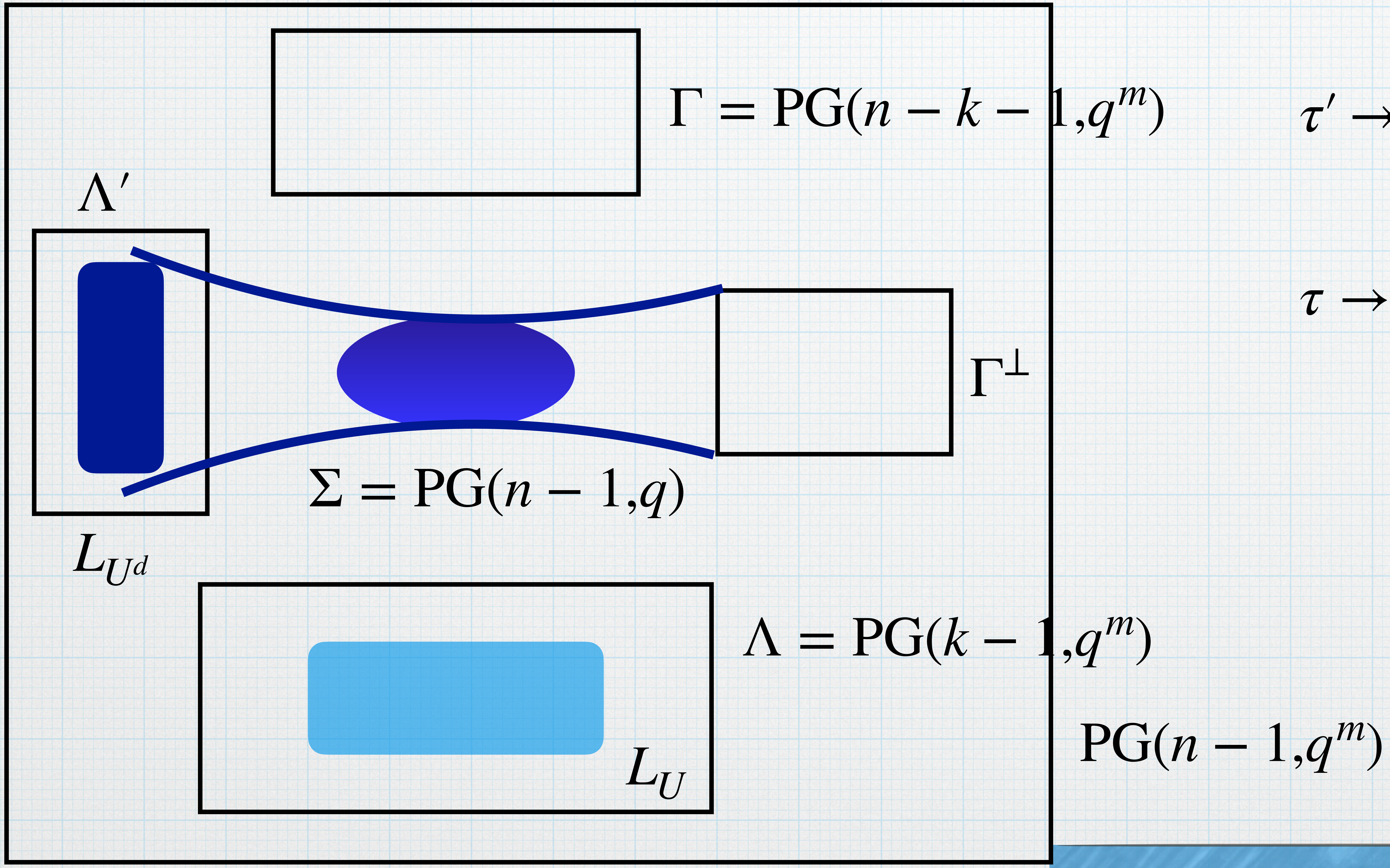

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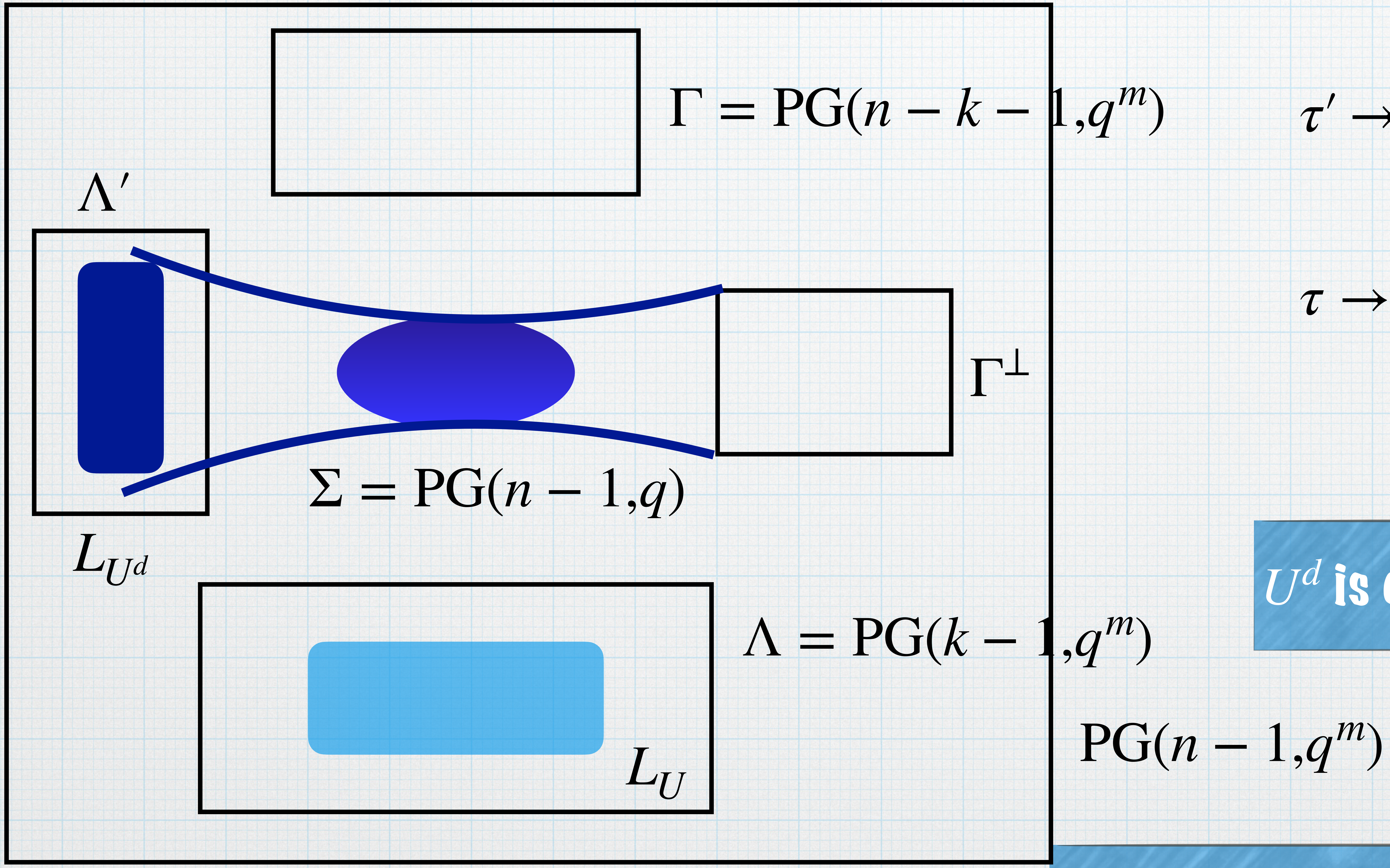

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U^d is called the Delsarte dual of U

Csajbók, B., Marino, G., Polverino, O., & FZ (2021). Generalising the scattered property of subspaces. Combinatorica, 41(2), 237-262.

Delsarte duality

Borello M., Polverino O. And FZ. Delsarte duality on subspaces and applications to rank-metric codes and q-matroids. Ongoing project.

$$n = n_1 + n_2, \;\; k = k_1 + k_2$$

Delsarte duality

Borello M., Polverino O. And FZ. Delsarte duality on subspaces and applications to rank-metric codes and q-matroids. Ongoing project.

$(\Lambda_{k-1,k_1,k_2}, k-1)$**-evasive**

$(\Lambda_{n-k-1,n_1-k_1,n_2-k_2}, n-k-1)$**-evasive**

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Delsarte duality

Borello M., Polverino O. And FZ. Delsarte duality on subspaces and applications to rank-metric codes and q-matroids. Ongoing project.

Theorem (Borello, Polverino and FZ)

$\mathcal{U}_{k_1,n_1}(q) \oplus \ldots \oplus \mathcal{U}_{k_t,n_t}(q)$ is $\mathbb{F}_{q^m}$-representable $\Leftrightarrow$ $\mathcal{U}_{n_1-k_1,n_1}(q) \oplus \ldots \oplus \mathcal{U}_{n_t-k_t,n_t}(q)$ is $\mathbb{F}_{q^m}$-representable

In conclusion...

* Representability of the direct sum of uniform q-matroids

* Open: to find the smallest extension field required for the representability of the direct sum of uniform q-matroids

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→ More on the existence of $(\Lambda_{k-1,k_1,k_2}, k-1)$-evasive

In conclusion...

* Representability of the direct sum of uniform q-matroids
* Open: to find the smallest extension field required for the representability of the direct sum of uniform q-matroids

$\Rightarrow$ More on the existence of $(\Lambda_{k-1,k_1,k_2}, k-1)$-evasive

* Open: what about the direct sum of other classes of representable q-matroids?

OpeRa 2026 - Bordeaux

Open Problems on Rank-Metric Codes

February 23-27, 2026

Save the date!

Invited speakers

- Xavier Caruso (Universitè de Bordeaux - France)
- Alain Couvreur (INRIA Saclay - France)
- Anna-Lena Horlemann (University of St. Gallen - Switzerland)
- Relinde Jurrius (Netherlands Defence Academy - the Netherlands)
- Alberto Ravagnani (Eindhoven University of Technology - the Netherlands)
- John Sheekey (University College Dublin - Ireland)

Organizers

- Gianira Alfarano
- Elena Berardini
- Martino Borello
- Ferdinando Zullo

Thanks for your attention!