Exploring Quasi-Hermitian Varieties: Properties and Applications

Angela Aguglia
Dipartimento di Meccanica, Matematica e Management
Politecnico di Bari, Italy

based on joint works with
Cossidente, Giuzzi, Korchmáros, Longobardi, Montinaro, Siconolfi

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Quasi-Polar Spaces

Definition (Schillewaert, Van de Voorde (2022))

A **quasi-polar space** is a set of points $\mathcal{S}$ in $\mathrm{PG}(r, q)$, where $r \geq 2$ and q is a prime power, such that the intersection sizes with hyperplanes match those of a non-degenerate classical polar space $\mathscr{P}$ embedded in $\mathrm{PG}(r, q)$.

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Reference:

- B. Segre, Sulle ovali nei piani lineari finiti, *Rendiconti dell'Accademia Nazionale dei Lincei*, (1954).

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The analogous concept for Hermitian varieties, that is *quasi-Hermitian varieties*, was formally introduced in 2010 by **De Winter and Schillewaert**.

The Hermitian Case

A **non-singular Hermitian variety** $\mathcal{H}(r, q^2)$ in the projective space $\mathrm{PG}(r, q^2)$ is defined as the set of absolute points of a non-degenerate unitary polarity ρ.

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Property

A non-singular Hermitian variety $\mathcal{H}(r, q^2)$ is a hypersurface with equation:

$$(X_0^q, \ldots, X_r^q) H (X_0, \ldots, X_r)^T = 0,$$

where H is a non-singular Hermitian $(r+1) \times (r+1)$ matrix.

Projective Equivalence

Any non-singular Hermitian variety in $\mathrm{PG}(r, q^2)$ can be mapped to any other non-singular Hermitian variety in $\mathrm{PG}(r, q^2)$ via a projectivity.

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In the plane, the non-singular Hermitian curve $\mathcal{H}(2, q^2)$ is also known as the **classical or Hermitian unital**.

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In the plane, the non-singular Hermitian curve $\mathcal{H}(2, q^2)$ is also known as the **classical or Hermitian unital**.

A **unital embedded** in $\mathrm{PG}(2, q^2)$ is a set of $q^3 + 1$ $(= |\mathcal{H}(2, q^2)|)$ points such that every line of the plane intersects it in either 1 or $q + 1$ points.

Definition (De Winter, Schillewaert (2010))

A point set $\mathcal{S}$ of $\mathrm{PG}(r, q^2)$ is a **quasi-Hermitian variety** if it meets each hyperplane in either

$$|\mathcal{H}(r-1, q^2)| = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1}, \text{ or}$$

$$|P_0\mathcal{H}(r-2, q^2)| = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} + (-1)^{r-1}q^{r-1}$$

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points.

$\mathcal{H}(r, q^2)$ is a quasi-Hermitian variety, called the *classical quasi-Hermitian variety*.

Quasi-Hermitian Varieties as Two-Character Sets

Property

A *quasi-Hermitian variety* in the projective space $\mathrm{PG}(r, q^2)$ is a **two-character set**, meaning a point set with exactly two possible intersection sizes with hyperplanes.

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Why It Matters

Two-character sets have wide-ranging applications:

- They give rise to **strongly regular graphs**.
- They generate **two-weight linear codes**.

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Key references:

- R. Delsarte, Weights of linear codes and strongly regular normed spaces, *Discrete Math.*, **3** (1972)
- R. Calderbank, W. Kantor, The geometry of two-weight codes, *The Bulletin of the London Mathematical Society*, **18** (1986)

Cardinality of a quasi-Hermitian variety

Theorem 1 (Schillewaert, Van de Voorde (2022))

Let $\mathcal{S}$ be a quasi-Hermitian variety in $\mathrm{PG}(r, q^2)$ with $r \geq 3$, then:

$$|\mathcal{S}| = |\mathcal{H}(r, q^2)|.$$

If $\mathcal{S} \subseteq \mathrm{PG}(2, q^2)$ is a point set such that every line intersects $\mathcal{S}$ in either 1 or $q+1$ points, then:

$$|\mathcal{S}| \in \{q^2 + q + 1,\ q^3 + 1\}.$$

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Perspective

We interpret a quasi-Hermitian variety $\mathcal{S}$ as higher-dimensional generalization of a unital. Thus, $|\mathcal{S}| = |\mathcal{H}(r, q^2)|$ for any $r \geq 2$.

Contents

Known Constructions of Q-H varieties

All Dimensions $r > 2$

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Three-Dimensional Constructions

- Lia, Lavrauw, Pavese (2024),
- Lia, Sheekey (2024).

BM quasi-Hermitian Variety Construction

Fix a projective frame in $\mathrm{PG}(r, q^2)$ with homogeneous coordinates $(X_0, X_1, \ldots, X_r)$.

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Consider the affine space $\mathrm{AG}(r, q^2)$ whose infinite hyperplane Σ_∞ has equation $X_0 = 0$.

Then $\mathrm{AG}(r, q^2)$ has affine coordinates $(x_1, x_2, \ldots, x_r)$ where $x_i = X_i/X_0$ for $i \in \{1, \ldots, r\}$.

Take $a \in \mathrm{GF}(q^2)$ and $b \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ and consider the projective variety $\mathcal{B}_{a,b}$ of equation

$$X_r^q X_0^q - X_r X_0^{2q-1} + a^q(X_1^{2q} + \ldots + X_{r-1}^{2q}) - a(X_1^2 + \ldots + X_n^2)X_0^{2q-2} \\ = (b^q - b)(X_1^{q+1} + \ldots + X_{r-1}^{q+1})X_0^{q-1}. \quad (1)$$

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Let $\mathcal{F} \subset \Sigma_\infty$ be the Hermitian cone

$$\mathcal{F} := \{(0, x_1, \ldots, x_r) | x_1^{q+1} + \ldots + x_{r-1}^{q+1} = 0\}.$$

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If $a = 0$ then $\mathcal{B}_{a,b}$ is a non-singular Hermitian variety.

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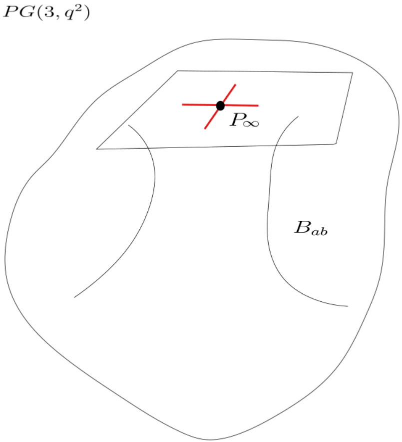
4. r is odd and q is even.

If $r = 2$ then $\mathcal{B}_{a,b}$ is a non-classical Buekenhout-Metz unital.

Case $r = 3$ and q odd

$\mathcal{B}_{a,b} \cap \Sigma_\infty = \ell_1 \cup \ell_2$ where $\ell_1 : X_1 - \nu X_2 = 0 = X_0$, $\ell_2 : X_1 + \nu X_2 = 0 = X_0$ and $\nu \in \mathrm{GF}(q^2) : \nu^2 = -1$.



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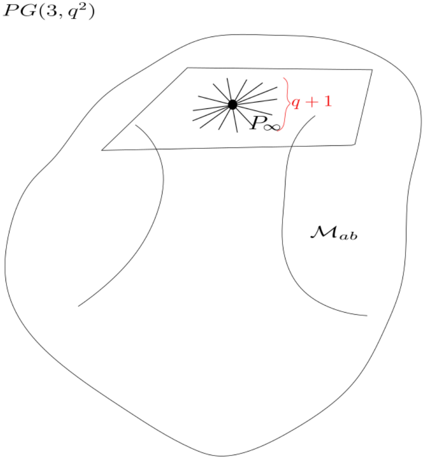

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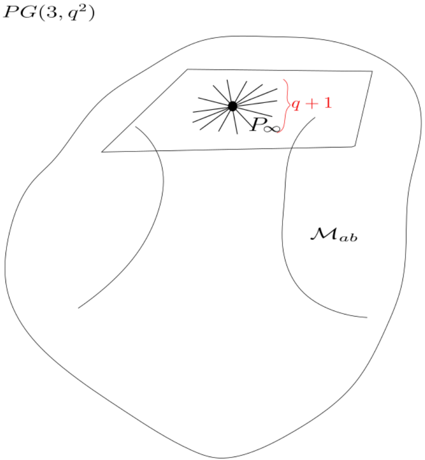

For any $r \geq 2$, we define $\mathcal{M}_{\mathbf{a,b}} := (\mathcal{B}_{a,b} \setminus \Sigma_\infty) \cup \mathcal{F}$

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Theorem 2 (A., Cossidente, Korchmáros (2012))

The set $\mathcal{M}_{a,b}$ consisting of the affine points of $\mathcal{B}_{a,b}$ plus the infinite points of $\mathcal{F}$ is a non–classical quasi-Hermitian variety of $\mathrm{PG}(r, q^2)$, $r \geq 2$.

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Crucial Tool: A Non-Standard Model of $\mathrm{PG}(r, q^2)$

Fix $a \in \mathrm{GF}(q^2)^*$.

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Define the quadric $\mathcal{Q}_a(m, d)$ for $m = (m_1, \ldots, m_{r-1}) \in \mathrm{GF}(q^2)^{r-1}$ and $d \in \mathrm{GF}(q^2)$ as the set of points satisfying:

$$x_r = a(x_1^2 + \ldots + x_{r-1}^2) + m_1x_1 + \ldots + m_{r-1}x_{r-1} + d.$$

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Consider the pair $(\mathcal{P}, \Sigma)$, where:

- **Points** $P \in \mathcal{P}$: all points of the affine geometry $\mathrm{AG}(r, q^2)$;
- **Hyperplanes** $\pi \in \Sigma$:
    - all affine hyperplanes passing through the point at infinity $P_\infty = (0, 0, \ldots, 0, 1)$;
    - all quadrics of the form $\mathcal{Q}_a(m, d)$.

Lemma 3 (A., Cossidente, Korchmáros, (2012))

For every non-zero $a \in \mathrm{GF}(q^2)$, $(\mathcal{P}, \Sigma)$ *defines an incidence structure* Π_a *isomorphic to* $\mathrm{AG}(r, q^2)$.

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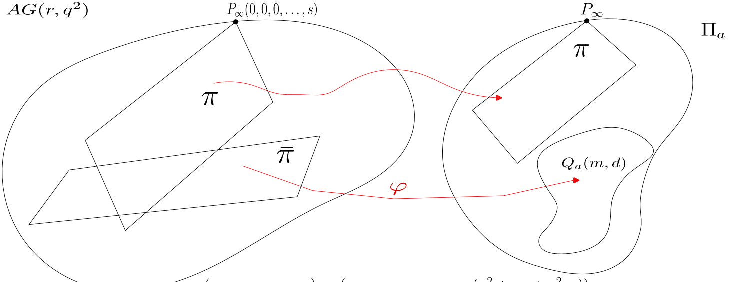

$$\varphi : (x_1, \dots, x_{r-1}, x_r) \to (x_1, \dots, x_{r-1}, x_r - a(x_1^2 + \dots + x_{r-1}^2))$$

$$\pi : m_1x_1 + \dots + m_{r-1}x_{r-1} + d = 0$$

$$Q_a(m,d) : x_r = a(x_1^2 + \dots + x_{r-1}^2) + m_1x_1 + \dots + m_{r-1}x_{r-1} + d$$

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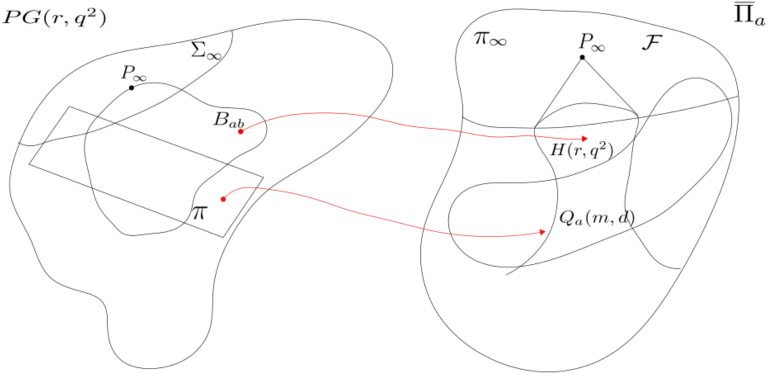

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Now, let $\mathcal{V}_\varepsilon^r$ be the variety of $\mathrm{PG}(r, q^2)$ represented by:

$$x_r^q + x_r = \Gamma_\varepsilon(x_1) + \ldots + \Gamma_\varepsilon(x_{r-1}) \tag{2}$$

where

$$\Gamma_\varepsilon(x) = [x + (x^q + x)\varepsilon]^{\sigma+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\varepsilon + x^{q+1} + x^2.$$

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We define

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where

$$\mathcal{F} = \{(0, X_1, \ldots, X_r) | \quad X_1^{q+1} + \ldots + X_{r-1}^{q+1} = 0\}.$$

Theorem [A. (2013)]

$\mathcal{H}_\varepsilon^r$ is a non-classical quasi-Hermitian variety in $\mathrm{PG}(r, q^2)$ which, for $r = 2$, corresponds to a Buekenhout–Tits unital.

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Automorphism 2-Group Result [A.(2013)]

The elementary abelian 2-group E of order q^r, generated by collineations with matrices of the form:

$$\begin{pmatrix} 1 & \gamma_1\varepsilon & \gamma_2\varepsilon & \cdots & \gamma_{r-1}\varepsilon & \gamma_r+(\sum_{i=1}^{r-1}\gamma_i)^\sigma\varepsilon \\ 0 & 1 & 0 & \cdots & 0 & \gamma_1+\gamma_1\varepsilon \\ 0 & 0 & 1 & \cdots & 0 & \gamma_2+\gamma_2\varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \gamma_{r-1}+\gamma_{r-1}\varepsilon \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with $\gamma_i\in\mathbb{F}_q$ for $i=1,\ldots,r$, is a subgroup of $\mathrm{Aut}(\mathcal{H}_\varepsilon^r)$.

A Classical Problem

Having only a few intersection numbers with hyperplanes is a strong combinatorial property however, this condition alone is not sufficient to characterize Hermitian varieties.

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Problem

Can we find a characterization of Hermitian varieties among quasi-Hermitian ones?

Some Known Characterizations

Theorem 4 (De Winter, Schillewaert (2010))

A quasi-Hermitian variety of $\mathrm{PG}(r, q^2)$*,* $r \geq 3$ *is classical if it has the same intersection numbers with respect to spaces of codimension 2 as a non-singular Hermitian variety.*

Some Known Characterizations

Theorem 4 (De Winter, Schillewaert (2010))

A quasi-Hermitian variety of $\mathrm{PG}(r, q^2)$, $r \geq 3$ *is classical if it has the same intersection numbers with respect to spaces of codimension 2 as a non-singular Hermitian variety.*

Theorem 5 (A., Bartoli, Storme, Weiner (2018))

A quasi-Hermitian variety of $\mathrm{PG}(r, q^2)$, *with* $r = 3$ *and* $q = p^h > 4$, *or* $r \geq 4$, $q = p > 4$, *or* $r \geq 4$, $q = p^2$, $p > 3$ *prime, is classical if and only if it is in the* $\mathbb{F}_p$*-code spanned by the hyperplanes of* $\mathrm{PG}(r, q^2)$.

Theorem 6 (Napolitano (2023))

Let $\mathcal{H}$ *be a set of points in* $\mathrm{PG}(3, q^2)$*, with* $q \neq 2$*, such that:*

- $\mathcal{H}$ *has the same size as the Hermitian surface;*
- $\mathcal{H}$ *contains no plane;*
- *every line is either fully contained in* $\mathcal{H}$ *or intersects* $\mathcal{H}$ *in at most* $q + 1$ *points;*
- *every plane intersects* $\mathcal{H}$ *in at least* $q^3 + 1$ *points.*

Then $\mathcal{H}$ *is a* **quasi-Hermitian variety**.

Moreover, if there is no external line, then $\mathcal{H}$ *is a* **Hermitian surface**.

Research Question and Approach

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Problem

Can we characterize BM and BT quasi-Hermitian varieties based on their incidence properties or their automorphism groups?

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Can we characterize BM and BT quasi-Hermitian varieties based on their incidence properties or their automorphism groups?

Our Approach

- Classify these varieties up to projective equivalence.
- Determine their full automorphism groups.
- Derive group-theoretic characterizations.

Contents

Projective equivalence classes of $\mathcal{M}_{a,b}$.

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How many projectively inequivalent BM Q-H varieties can be obtained from varying the parameters (a, b)?

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How many projectively inequivalent BM Q-H varieties can be obtained from varying the parameters (a, b)?

Theorem 7 (Baker and Ebert (1982), Ebert (1993))

Let $q = p^n \geq 4$ be a prime power. Then the number of projectively inequivalent BM unitals of $\mathrm{PG}(2, q^2)$ is

$$\frac{1}{2n}\left[n_0 + \sum_{k|n} \Phi\left(\frac{2n}{k}\right) p^k\right],$$

where Φ is the Euler Φ-function and n_0 is the odd part of n if $p > 2$ or $n_0 = 0$ if $p = 2$.

Case $r = 3$

Theorem 8 (A., Giuzzi (2023))

Let $q = p^n$ with p an odd prime. Then the number of projectively inequivalent BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$ of $\mathrm{PG}(3, q^2)$ is

$$\frac{1}{n}\left(\sum_{k|n} \Phi\left(\frac{n}{k}\right) p^k\right) - 2,$$

where Φ is the Euler Φ-function.

Case $r = 3$

Theorem 8 (A., Giuzzi (2023))

Let $q = p^n$ with p an odd prime. Then the number of projectively inequivalent BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$ of $\mathrm{PG}(3, q^2)$ is

$$\frac{1}{n}\left(\sum_{k|n}\Phi\left(\frac{n}{k}\right)p^k\right) - 2,$$

where Φ is the Euler Φ-function.

Theorem 9 (A., Giuzzi, Montinaro, Siconolfi (2025))

All BM quasi-Hermitian varieties $\mathcal{M}_{a,b}$ of $\mathrm{PG}(3, q^2)$, q even are equivalent.

Crucial tools

Assume q odd and set:

$$\ell_1 : X_1 - \nu X_2 = 0 = X_0, \quad \ell_2 : X_1 + \nu X_2 = 0 = X_0$$

where $\nu \in \mathrm{GF}(q^2)$ such that $\nu^2 + 1 = 0$.

Theorem 10 (A., Giuzzi (2023))

Let $\mathcal{M}_{a,b}$ be a BM quasi-Hermitian variety of $\mathrm{PG}(3, q^2)$, with $q \equiv 1 \pmod 4$. Then,

- *through each affine point of $\mathcal{M}_{a,b}$ pass exactly two lines of $\mathcal{M}_{a,b}$;*
- *through each point at infinity on the union $\ell_1 \cup \ell_2$ there pass $q + 1$ lines of a pencil contained in $\mathcal{M}_{a,b}$;*
- *through each point at infinity not on $\ell_1 \cup \ell_2$ there passes only one line of $\mathcal{M}_{a,b}$.*

Crucial tools

Theorem 11 (A., Giuzzi (2023))

Let $\mathcal{M}_{a,b}$ *be a BM quasi-Hermitian variety of* $\mathrm{PG}(3,q^2)$, *with* $q \equiv 3 \pmod 4$. *Then,*

- *no line of* $\mathcal{M}_{a,b}$ *passes through any affine point of* $\mathcal{M}_{a,b}$*;*
- *through each point at infinity in* $(\mathcal{M}_{a,b} \cap \Sigma_\infty) \setminus \{P_\infty\}$ *there passes only one line of* $\mathcal{M}_{a,b}$.
- *through the point* $P_\infty(0,0,\dots,1)$ *there are* $q+1$ *lines contained in* $\mathcal{M}_{a,b}$

Suppose q even and set ℓ_∞: $X_0 = X_1 + X_2 = 0$.

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Theorem 12 (A., Giuzzi, Montinaro, Siconolfi (2025))

Let $\mathcal{M}_{a,b}$ be the BM quasi-Hermitian variety of $\mathrm{PG}(3, q^2)$, with q even. Then:

- *through each affine point of $\mathcal{M}_{a,b}$ there passes exactly one line of $\mathcal{M}_{a,b}$;*
- *through each point at infinity in $\mathcal{M}_{a,b} \cap \ell_\infty$ there pass $q+1$ lines of a pencil contained in $\mathcal{M}_{a,b}$.*

Automorphism groups in $\mathrm{PG}(3, q^2)$, q even prime power

Let ϕ_s, ψ_γ, μ_δ, and τ_e be the collineations associated with the following matrices, where:

- $s, e \in \mathrm{GF}(q)$,
- $\delta \in \mathrm{GF}(q)^*$,
- $\gamma = (\gamma_1, \gamma_2) \in \mathrm{GF}(q^2)^2$.

$$\phi_s : \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \psi_\gamma(a, b) : \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & a(\gamma_1^2 + \gamma_2^2) + b(\gamma_1^{q+1} + \gamma_2^{q+1}) \\ 0 & 1 & 0 & (b + b^q)\gamma_1^q \\ 0 & 0 & 1 & (b + b^q)\gamma_2^q \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mu_\delta : \mathrm{diag}(1, \delta, \delta, \delta^2), \quad \tau_e : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e+1 & e & 0 \\ 0 & e & e+1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 13 (A., Giuzzi, Montinaro, Siconolfi (2025))

The stabilizer of $\mathcal{M}_{a,b}$ in $PGL_4(q^2)$, q even, is the group

$$G(a,b) = \left\langle \phi_s, \psi_\gamma(a,b), \tau_e, \mu_\delta : \gamma \in \mathrm{GF}(q^2)^2, s, e, \delta \in \mathrm{GF}(q), \delta \neq 0 \right\rangle$$

of order $q^6(q-1)$.

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Theorem 14 (A., Giuzzi, Montinaro, Siconolfi (2025))

Let $\sigma \in P\Gamma L_4(q^2)$ be induced by a generator of $\mathrm{Aut}(\mathrm{GF}(q^2))$, q even, and let $\beta \in P\Gamma L_4(q^2)$ map $\mathcal{M}_{1,\epsilon}$ with $\mathrm{tr}_{q^2/q}(\epsilon) = 1$ onto $\mathcal{M}_{a,b}$. Then the stabilizer in $P\Gamma L_4(q^2)$ of $\mathcal{M}_{a,b}$ is

$$\Gamma(a,b) = \left\langle \phi_s, \psi_\gamma(a,b), \tau_e, \mu_\delta, \sigma^\beta : \gamma \in \mathrm{GF}(q^2)^2, s, e, \delta \in \mathrm{GF}(q), \delta \neq 0 \right\rangle,$$

and its order is $q^6(q-1)\log_2 q$.

Projective Equivalence Classes of $\mathcal{H}_{\varepsilon}^{r}$

Case $r = 2$**:** The BT quasi-Hermitian varieties coincide with the BT unitals.

Projective Equivalence Classes of $\mathcal{H}_{\varepsilon}^{r}$

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Result: All BT unitals are equivalent under the action of $P\Gamma L(3, q^2)$, as proven by **J. Faulkner** and **G. Van de Voorde** (2025), resolving an open problem posed by Barwick and Ebert in their book *Unitals in Projective Planes* (2008).

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Problem

What occurs in higher-dimensional spaces?

Theorem 15 (A., Montinaro (submitted))

Let $\varepsilon_1, \varepsilon_2 \in \mathbb{F}_{q^2} : \varepsilon_i^2 + \varepsilon_i = \delta_i$ *with* $T(\delta_i) = 1$, $i = 1, 2$. *Assume* $\alpha = \varepsilon_2 - \varepsilon_1$, $B = (\frac{\alpha}{\delta_1})^{\sigma/2}$, $\rho = \left(\frac{\delta_1}{\delta_2}\right)^{\sigma/2+1}$ *and* $r \geq 2$. *The projectivity* ξ *represented by the* $(r+1) \times (r+1)$ *matrix* A *below, maps* $\mathcal{H}^r_{\varepsilon_1}$ *onto* $\mathcal{H}^r_{\varepsilon_2}$

$$A = \begin{pmatrix} 1 & B\rho\frac{\varepsilon_2}{\varepsilon_1} & B\rho\frac{\varepsilon_2}{\varepsilon_1} & \cdots & B\rho\frac{\varepsilon_2}{\varepsilon_1} & (r-1)\rho^2 B^{\sigma+2}\frac{\varepsilon_2^{q+2}}{\varepsilon_1^{q+1}} \\ 0 & \frac{\rho\varepsilon_2}{\varepsilon_1} & 0 & \cdots & 0 & B\rho^2\frac{\varepsilon_2^{q+1}}{\varepsilon_1^{q+1}} \\ 0 & 0 & \frac{\rho\varepsilon_2}{\varepsilon_1} & \cdots & 0 & B\rho^2\frac{\varepsilon_2^{q+1}}{\varepsilon_1^{q+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\rho\varepsilon_2}{\varepsilon_1} & B\rho^2\frac{\varepsilon_2^{q+1}}{\varepsilon_1^{q+1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\rho^2\varepsilon_2^{q+1}}{\varepsilon_1^{q+1}} \end{pmatrix}$$

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Lemma 16 (A., Montinaro (submitted))

A Sylow 2-subgroup of Aut$(\mathcal{H}^r_\varepsilon)$ *fixes a unique incident point-hyperplane pair of* $\mathrm{PG}(r, q^2)$.

Collineation group of $\mathcal{H}_\varepsilon^3$

Theorem 17 (A., Montinaro (subm.))

The following hold:

1. $Aut(\mathcal{H}_\varepsilon^3) \cap PGL_4(q^2) = E\,\langle\vartheta\rangle$, *with* $\vartheta : (x_0, x_1, x_2, x_3) \to (x_0, x_2, x_1, x_3)$, *is a group of order* $2q^3$*;*
2. $Aut(\mathcal{H}_\varepsilon^3)$ *preserves the triple* $(P_\infty, \ell_\infty, \Sigma_\infty)$*.*

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2. $Aut(\mathcal{H}_\varepsilon^3)$ *preserves the triple* $(P_\infty, \ell_\infty, \Sigma_\infty)$.

Theorem 18 (A., Montinaro (subm.))

$Aut(\mathcal{H}_\varepsilon^3) = E\langle\vartheta, \phi\rangle$, *with* $\vartheta : (x_0, x_1, x_2, x_3) \to (x_0, x_2, x_1, x_3)$ *and*

$$\phi : (x_0, x_1, x_2, x_3) \to (x_0^2, x_1^2, x_2^2, x_3^2) \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & \delta^{\frac{\sigma}{2}}\varepsilon^q & 0 & \delta^{\frac{\sigma}{2}}\varepsilon^q \\ 0 & 0 & \delta^{\frac{\sigma}{2}}\varepsilon^q & \delta^{\frac{\sigma}{2}}\varepsilon^q \\ 0 & 0 & 0 & \delta^{\sigma+1} \end{pmatrix},$$

is a group of order $4eq^3$.

Tools: some geometric properties

Set $\ell_\infty : X_0 = X_1 + X_2 = 0$.

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Theorem 19 (A., Montinaro (submitted))

Let $\mathcal{H}^3_\varepsilon$ be the BT quasi-Hermitian variety of $\mathrm{PG}(3, q^2)$. Then the following statements hold:

1. *each affine line contained in $\mathcal{H}^3_\varepsilon$ intersects ℓ_∞ at one among the points $L_\infty^{\varepsilon^q\alpha} = (0, 1, 1, \varepsilon^q\alpha)$, where $\alpha \in \mathrm{GF}(q)$;*
2. *through each point $L_\infty^{\varepsilon^q\alpha}$, there exist exactly $q + 1$ coplanar lines contained in $\mathcal{H}^3_\varepsilon$, one of which is ℓ_∞. Each such set of $q + 1$ lines forms a Hermitian cone $\Pi_0\mathcal{H}(1, q^2)$.*

Contents

Construction of MDS codes

Let $G_1(x), \ldots, G_N(x)$ be N **multivariate polynomials** over $\mathrm{GF}(q)$ and $\mathcal{W} \subset \mathrm{GF}(q)^{n+1}$.

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Here, the evaluation code $\mathcal{C} := \mathcal{C}(G_1, \ldots, G_N; \mathcal{W})$ defined by $G_1, \ldots, G_N$ over a set $\mathcal{W}$ is the image of the map

$$ev_{G_1,\ldots,G_N} : \begin{cases} \mathcal{W} \to \mathrm{GF}(q)^N \\ x \to (G_1(x), \ldots, G_N(x)). \end{cases}$$

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Assume that $\mathcal{C}$ has q^t codewords and Hamming distance d.

The Singleton Bound

The Singleton bound

It is known that for a code C over $\mathrm{GF}(q)$ with parameters $[N, q^t, d]$, the following holds:

$$q^t \leq q^{N-d+1}$$

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If the evaluation code $\mathcal{C}$ is MDS, then any $t = N - d + 1$ of the varieties $V(G_i) : G_i = 0$, for $i = 1, \ldots, N$, must intersect in exactly one point in the ambient space $\mathcal{W}$.

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Define $\mathcal{W} := \mathrm{GF}(q^2) \times \mathrm{GF}(q^2) \times T$, where T is a transversal of $\mathrm{GF}(q)$ viewed as an additive subgroup of $\mathrm{GF}(q^2)$.

Determinantal Condition

Consider the subset Ω of $\mathrm{GF}(q^2)^2$ such that for each $(\omega_1^i, \omega_2^i) \in \Omega$ with $i \geq 5$, the following condition holds:

$$\det \begin{pmatrix} 1 & \omega_1^1 & \omega_2^1 & (\omega_1^1)^q & (\omega_2^1)^q \\ 1 & \omega_1^2 & \omega_2^2 & (\omega_1^2)^q & (\omega_2^2)^q \\ 1 & \omega_1^3 & \omega_2^3 & (\omega_1^3)^q & (\omega_2^3)^q \\ 1 & \omega_1^4 & \omega_2^4 & (\omega_1^4)^q & (\omega_2^4)^q \\ 1 & \omega_1^5 & \omega_2^5 & (\omega_1^5)^q & (\omega_2^5)^q \end{pmatrix} \neq 0 \qquad \text{(DetCond)}$$

Fix a basis $(1, \epsilon)$ of $\mathrm{GF}(q^2)$ regarded as a vector space over $\mathrm{GF}(q)$, with $\epsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$: $\mathrm{tr}_{q^2/q}(\epsilon) = 0$ for q odd or $\mathrm{tr}_{q^2/q}(\epsilon) = 1$ for q even.

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Write $\omega_i^j = \omega_{i,0}^j + \omega_{i,1}^j \epsilon$ for all $i = 1, 2$ and $j = 1, \ldots, 5$.

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Then (DetCond) becomes

$$\det \begin{pmatrix} 1 & \omega_{1,0}^1 & \omega_{2,0}^1 & \omega_{1,1}^1 & \omega_{2,1}^1 \\ 1 & \omega_{1,0}^2 & \omega_{2,0}^2 & \omega_{1,1}^2 & \omega_{2,1}^2 \\ 1 & \omega_{1,0}^3 & \omega_{2,0}^3 & \omega_{1,1}^3 & \omega_{2,1}^3 \\ 1 & \omega_{1,0}^4 & \omega_{2,0}^4 & \omega_{1,1}^4 & \omega_{2,1}^4 \\ 1 & \omega_{1,0}^5 & \omega_{2,0}^5 & \omega_{1,1}^5 & \omega_{2,1}^5 \end{pmatrix} \neq 0 \qquad \text{(DetCondq)}$$

for any choice of five elements in Ω.

Condition (DetCondq) states that the rows of the matrix represent the coordinates of points lying on an arc in $\mathrm{AG}(4, q)$.

Arcs in Projective Space

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An arc in $\mathrm{PG}(4, q)$, that is a set of points such that any five span the entire space, has maximum size $q + 1$.

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Therefore, we have the bound: $|\Omega| \leq q$.

Theorem 20 (Ball, Lavrauw (2019))

The only $(q + 1)$-arcs in $\mathrm{PG}(4, q)$ for $q > 13$ odd or $q \geq 8$ even, are projectively equivalent to the normal rational curve:

$$\Gamma_4 = \{(1, t, t^2, t^3, t^4) \mid t \in \mathrm{GF}(q)\} \cup \{(0, 0, 0, 0, 1)\}.$$

Defining the Forms F_i

We can set

$$\Omega := \{(t + \varepsilon t^2, t^3 + \varepsilon t^4) \colon t \in \mathrm{GF}(q)\}$$

so that it corresponds to an arc in $AG(4, q)$.

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so that it corresponds to an arc in $AG(4, q)$.

Now, consider the following forms

$$F_i(X_0, X_1, X_2, X_3) = (b - b^q) X_0^{q-1}(X_1^{q+1} + X_2^{q+1}) +$$
$$X_0^q X_3^q - X_3 X_0^{2q-1} + a^q(X_1^{2q} + X_2^{2q}) - a X_0^{2q-2}(X_1^2 + X_2^2) +$$
$$[2a^q(\omega_1^i)^q - (b^q - b)\omega_1^i] X_0^q X_1^q + [2a^q(\omega_2^i)^q - (b^q - b)\omega_2^i] X_0^q X_2^q +$$
$$[2a\omega_1^i + (b - b^q)(\omega_1^i)^q] X_0^{2q-1} X_1 + [2a\omega_2^i + (b - b^q)(\omega_2^i)^q] X_0^{2q-1} X_2$$

where $(\omega_1^i, \omega_2^i) \in \Omega$ for $i = 1, \dots, q$.

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$$\mathcal{C}(F_1, \ldots, F_q; \mathcal{W}) =$$

$$\{(F_1(1, x, y, z), F_2(1, x, y, z), \ldots, F_q(1, x, y, z)) | (x, y, z) \in \mathcal{W}\}.$$

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Theorem 21 (A., Giuzzi, Siconolfi (2025))

Let $q > 13$. The code $\mathcal{C}(F_1, \dots, F_q; \mathcal{W})$ is a $\mathrm{GF}(q)$-linear $[q, 5, q-4]$-MDS code over T_0.

Some Equivalent Codes

Write $\varepsilon^q + \varepsilon = a_0$, so $\mathrm{tr}_{q^2/q}(x_0 + \varepsilon x_1) = 2x_0 + a_0 x_1$, for $x_0, x_1 \in \mathrm{GF}(q)$.

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So $T_0 = \{x \in \mathrm{GF}(q^2) | \mathrm{tr}_{q^2/q}(x) = 0\} = \{(a_0 - 2\varepsilon)x_0 | x_0 \in \mathrm{GF}(q)\}$.

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Theorem 22 (A., Giuzzi, Siconolfi (2025))

The code $\mathcal{C}' := (\theta^{-1})\mathcal{C}(F_1, F_1, \dots, F_q; \mathcal{W})$ is equivalent to a q-ary Reed-Solomon code. In particular, it can be further extended to a $[q+1, 5, q-3]_q$ Reed-Solomon code.

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The extended Reed-Solomon code is obtained by $\mathcal{C}'$ adding to each codeword the component $F_{q+1}(1, x, y, z)$ where

$$F_{q+1}(X_0, X_1, X_2, X_3) = (b^q - b)X_2^q X_0 + 2aX_2 X_0^q$$

.

Non-degeneracy and Minimal Codes

Let $\mathcal{C}$ be a q-ary linear $[n, r, d]$ code. The *support* of a codeword $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ is: $\mathrm{supp}(\boldsymbol{c}) := \{i : c_i \neq 0\}$.

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A codeword $\boldsymbol{c} \in \mathcal{C}$ is *minimal* if for every non-zero codeword $\boldsymbol{c}' \in \mathcal{C}$ such that: $\mathrm{supp}(\boldsymbol{c}') \subseteq \mathrm{supp}(\boldsymbol{c})$, we have $\boldsymbol{c}' \in \langle \boldsymbol{c} \rangle$.

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References:

- Alfarano, Borello, Neri (2022)
- Scotti (2024)
- Alon, Bishnoi, Das, Neri (2024)

Code $\rightarrow$ Projective System

Let $\mathcal{C}$ be a non-degenerate q-ary linear $[n, r, d]$ code. Its *projective system* is the (multi)set Ω of points in $\mathrm{PG}(r-1, q)$ corresponding to the columns of any generator matrix of $\mathcal{C}$.

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Projective System $\rightarrow$ Code

Any (multi)set Ω of points in $\mathrm{PG}(r-1, q)$ defines a code $\mathcal{C}(\Omega)$ whose generator matrix has as columns the coordinates of the points in Ω.

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Projective System $\to$ Code

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Minimum Distance

For a code $\mathcal{C} = \mathcal{C}(\Omega)$, the minimum distance is given by:

$$d = |\Omega| - \max\{|\Omega \cap \Pi| : \Pi \text{ is a hyperplane of } \mathrm{PG}(\langle\Omega\rangle)\}.$$

Characterization of Minimal Codes

Theorem 23 (Alfarano, Borello, Neri (2022))

Let Ω *be a set of points in* $\mathrm{PG}(r,q)$ *such that* $\langle \Omega \rangle = \mathrm{PG}(r,q)$. *Then the code* $\mathcal{C}(\Omega)$ *is minimal if and only if for every hyperplane* Π *of* $\mathrm{PG}(r,q)$,

$$\langle \Pi \cap \Omega \rangle = \Pi,$$

i.e., Ω *is a cutting set.*

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Sufficient Condition (Ashikhmin–Barg)(1994)

A q-ary linear code $\mathcal{C}$ is minimal if $\frac{w_{\min}}{w_{\max}} > \frac{q-1}{q}$, where $w_{\min}$ and $w_{\max}$ are the minimum and maximum weights of non-zero codewords.

Minimality of the Code Associated with Q-H Varieties

If $\mathcal{H}$ is a quasi-Hermitian variety in $\mathrm{PG}(r, q^2)$, then the associated code $\mathcal{C}(\mathcal{H})$ has exactly **two distinct weights**.

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Theorem 24 (A., Giuzzi, Ceria (2022))

Let $\mathcal{H}$ be a quasi-Hermitian variety in $\mathrm{PG}(r, q^2)$. Then, the code $\mathcal{C}(\mathcal{H})$ is minimal.

Codes from $\mathcal{V}_\varepsilon^r$

Let us consider the BT quasi-Hermitian variety:

$$\mathcal{H}_\varepsilon^r = (\mathcal{V}_\varepsilon^r \setminus \Sigma_\infty) \cup \mathcal{F}.$$

What can be said about the code $\mathcal{C}_\varepsilon^r := \mathcal{C}(\mathcal{V}_\varepsilon^r)$?

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Key Properties

- It is a **few-weight code**.
- In some cases, the code is **minimal**

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Key Properties

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- In some cases, the code is **minimal**

Theorem 25 (A., Giuzzi, Longobardi, Siconolfi (submitted))

The linear code $\mathcal{C}^r_\varepsilon$ generated by the projective points of $\mathcal{V}^r_\varepsilon$ in $\mathrm{PG}(r, q^2)$ is a $(r+1)$-dimensional minimal code for $r = 3$ and $e \equiv 3 \pmod 4$ or $r \geq 4$ and any odd integer e.

Key Lemma (A., Giuzzi, Longobardi, Siconolfi (submitted))

Let $r \geq 2$ and $q = 2^e$ with $e \geq 3$ odd. Then the Fermat hypersurface $\mathcal{F}_n^r$ of degree $n = 2^{\frac{e-1}{2}} + 1$ in $\mathrm{PG}(r, q^2)$, defined by:

$$\mathcal{F}_n^r : X_0^n + X_1^n + \cdots + X_r^n = 0$$

spans the entire projective space: $\langle \mathcal{F}_n^r \rangle = \mathrm{PG}(r, q^2)$.

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The intersection $\mathcal{V}_\varepsilon^r \cap \Sigma_\infty$ is represented by:

$$X_0 = X_1^{2^{\frac{e-1}{2}}+1} + X_2^{2^{\frac{e-1}{2}}+1} + \cdots + X_{r-1}^{2^{\frac{e-1}{2}}+1} = 0.$$

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Remark

To determine the length and the weights of the projective linear code $\mathcal{C}_\varepsilon^r$, with $r \geq 3$, it is necessary to compute the number of $\mathrm{GF}(q^2)$-rational points of the Fermat hypersurface $\mathcal{F}_n^{r-2}$.

Proposition 26 (A., Giuzzi, Longobardi, Siconolfi (submitted))

The number N_{q^2} of $\mathrm{GF}(q^2)$-rational points of $\mathcal{F}_n^r$ in $\mathrm{PG}(r, q^2)$, $r \geq 2$, satisfies the following properties:

1. *if $e \equiv 1 \pmod 4$ then $N_{q^2} = \theta_{q^2}(r-1)$;*
2. *if $e \equiv 3 \pmod 4$ then*
$$N_{q^2} \leq (n-1)q^{2(r-1)} + nq^{2(r-2)} + \theta_{q^2}(r-3).$$
3. *if $e = 3$ and $r = 2$ then $N_{q^2} = (q+1)^2$.*

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3. *if $e = 3$ and $r = 2$ then $N_{q^2} = (q+1)^2$.*

Proposition 27 (A., Giuzzi, Longobardi, Siconolfi (submitted))

If $e \equiv 3 \pmod 4$, then the weights of the projective code $\mathcal{C}_\varepsilon^3 := \mathcal{C}(\mathcal{V}_\varepsilon^3)$ belong to the set

$$\{q^5, q^5 - q^3 + 3q^2, q^5 - q^3 + 2q^2, q^5 - q^3 + 3q^2 + q - 2, \\ q^5 - q^3 + 2q^2 + q - 2\}.$$

Theorem 28 (A., Giuzzi, Longobardi, Siconolfi (submitted))

The linear code $\mathcal{C}_\varepsilon^4 = \mathcal{C}(\mathcal{V}_\varepsilon^4)$ for $q = 2^e$ and $e > 1$ an odd integer is a 5-dimensional minimal code with the following parameters

1. *If* $e \equiv 1 \pmod 4$, *then* $\mathcal{C}_\varepsilon^4$ *has length* $|\mathcal{V}_\varepsilon^4| = q^7 + q^4 + q^2 + 1$ *and weights*

$$\{q^7, q^7 - q^5 + q^4 + q^3 - q^2, q^7 - q^5 + q^4 + q^2, \\ q^7 - q^5 + q^4, q^7 - q^5 + q^4 - q^2, q^7 - q^5 + q^4 - 4q^2\};$$

2. *If* $e = 3$, *then* $\mathcal{C}_\varepsilon^4$ *has length* $|\mathcal{V}_\varepsilon^4| = q^7 + q^4 + 2q^3 + q^2 + 1$ *and weights*

$$\{q^7, q^7 - q^5 + q^4 + 3q^3 - q^2, q^7 - q^5 + q^4 + 2q^3 + q^2, \\ q^7 - q^5 + q^4 + 2q^3, q^7 - q^5 + q^4 + 2q^3 - q^2, q^7 - q^5 + q^4 + 2q^3 - 2q^2\}.$$

Cutting Gap: Measuring Cutting Sets

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Definition 29 (Cutting Gap)

Let Ω be a non-empty set of points in $\mathrm{PG}(r,q)$, and let $0 \leq k \leq r$. The *k-th cutting gap* of Ω is defined as:

$$\tau_k(\Omega) = k - \min\left\{\dim\left(\langle \Pi \cap \Omega \rangle\right) : \dim(\Pi) = k\right\}.$$

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$$\tau_k(\Omega) = k - \min\left\{\dim\left(\langle \Pi \cap \Omega \rangle\right) : \dim(\Pi) = k\right\}.$$

Ω is a *k-fold strong blocking set* if and only if $\tau_k(\Omega) = 0$, according to the definitions in

- A.A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, Linear nonbinary covering codes and saturating sets in projective spaces, *Adv. Math. Commun.* **5**, (2011)

Cutting Sets and Their Properties

We also refer to this property as being *k-cutting*. Thus, Ω is a *k-cutting set* if $\tau_k(\Omega) = 0$.

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Proposition 30 (A. Giuzzi, Longobardi, Siconolfi, (submitted))

Let Ω be a k-cutting set in $\mathrm{PG}(r, q)$. Then Ω is also ℓ-cutting for all $k \leq \ell \leq r$.

Cutting Gap of Hermitian Varieties

Proposition 31 (A., Giuzzi, Longobardi, Siconolfi, (submitted))

Let $\mathcal{H}^r = \mathcal{H}(r, q^2)$ be a non-degenerate Hermitian variety in $\mathrm{PG}(r, q^2)$. Then the cutting gap satisfies:

$$\tau_{r-t}(\mathcal{H}^r) = \begin{cases} 0 & \text{if } t > \left\lfloor \frac{r}{2} \right\rfloor, \\ \\ 1 & \text{if } t \leq \left\lfloor \frac{r}{2} \right\rfloor. \end{cases}$$

Contents

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- **Characterization via cutting gaps:** Investigate whether the cutting gaps listed in the last previous proposition characterize Hermitian varieties among quasi-Hermitian ones.
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- **Sylow 2-subgroups:** Explore whether the property that any Sylow 2-subgroup of the collineation group of a BT Q-H variety fixes a unique incident point-hyperplane pair in $\mathrm{PG}(r, q^2)$ can be used to characterize the BT Q-H varieties.

Thank you

for your attention!

Grazie!

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