Block-transitive designs admitting multiple invariant partitions

Joint work with Seyed Hassan Alavi, Carmen Amarra, Ashraf Daneshkhah, Cheryl Praeger

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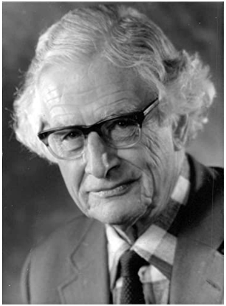
Statistical design of experiment

The theory of Design of Experiments was initiated by Ronald Fisher and Frank Yates in the early 1930's. They were motivated by questions of design of field experiments in agriculture.

Nowadays their work has many applications in various scientific fields.



Statistical design of experiment

Imagine you are a scientist needing to do an agricultural experiment to compare the yield of v different varieties of grain.

There could be an interaction between the environment (type of soil, rainfall, drainage, etc.) and the variety of grain which would alter the yields.

So you want to do experiments at a few different farms (called blocks) across the country.

Statistical design of experiment

Obvious solution: plant in v fields at each farm (complete blocks).
Downside: very costly. Perhaps the farms don't have that many fields available.

Other obvious solution: have $\binom{v}{2}$ farms (or a multiple of that) each with exactly two varieties.
Downside: very costly. Requires too many farms.

More practical solution: have a limited number of varieties planted at each farm (incomplete blocks). For each farm, you'll be able to compare yield between any two varieties planted there.

Statistical design of experiment

Ideally, to minimize the effects of chance due to incomplete blocks, you want to design the blocks so that the probability of two varieties being compared (i.e. are in the same block) is the same for all pairs. This property is called *balance* in the design.

This set up is called a Balanced Incomplete Block Design.

You can then use statistical techniques, such as Analysis of Variance (ANOVA), to reach conclusions about the experiment.

Definition

A balanced incomplete block design or 2-(v, k, λ) design is

- a set $\mathcal{P}$ of points (varieties) of size v,
- a set of $\mathcal{B}$ of blocks (farms) of size b,
- each block is a subset of size k of $\mathcal{P}$,
- each pair of distinct points lies on exactly λ blocks.

Incomplete $\Rightarrow k < v$

More than 2 varieties per block $\Rightarrow k > 2$

No repeated blocks

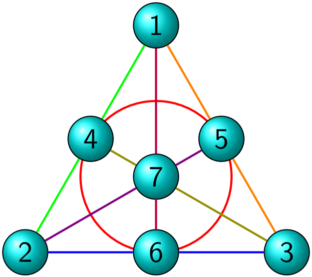
Example

Say you have $v = 7$ varieties, and $b = 7$ farms each with $k = 3$ fields available.

Here is a solution.

Farm	1	2	3	4	5	6	7
Varieties	1,2,4	2,3,6	3,5,1	3,4,7	2,5,7	1,6,7	4,5,6

Every pair of varieties compared once.
This is the Fano plane
2-$(7, 3, 1)$ design.

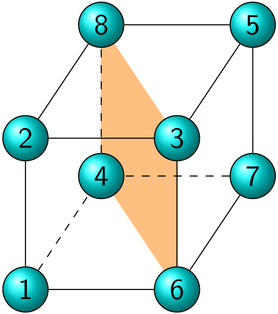


Example

Say you have $v = 8$ varieties, and $b = 14$ farms each with $k = 4$ fields available.
Here is a solution.

Farm	1	2	3	4	5	6	7
Varieties	1,2,4,8	1,3,7,8	2,3,5,8	3,4,6,8	4,5,7,8	1,5,6,8	2,6,7,8
Farm	8	9	10	11	12	13	14
Varieties	3,5,6,7	2,4,5,6	1,4,6,7	1,2,5,7	1,2,3,6	2,3,4,7	1,3,4,5

points=vertices of the cube,
blocks=planes.
Every pair of varieties compared 3 times.
2-$(8, 4, 3)$ design.



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- b and r can be recovered from the other parameters using the formulas
$$vr = bk \text{ and } \binom{v}{2}\lambda = b\binom{k}{2}$$
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(double count the number of incident (point,block) pairs, and the number of incident (point pairs, block) pairs)
- After rearranging
$$r = \frac{v-1}{k-1}\lambda \quad \text{and} \quad b = \frac{v(v-1)}{k(k-1)}\lambda$$

Properties of t-(v, k, λ) design

- Not all choices of parameters t, v, k, λ yield designs. The fact that b and r are integers gives us divisibility conditions $\Rightarrow$ admissible parameters.

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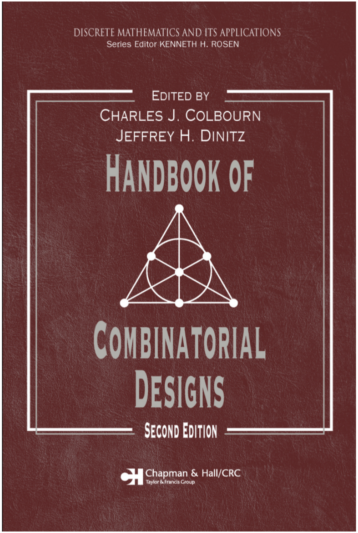
- Not all choices of parameters t, v, k, λ yield designs. The fact that b and r are integers gives us divisibility conditions $\Rightarrow$ admissible parameters.
- Even with admissible parameters, it is not obvious to determine if such a design exists
- or how many there are (up to isomorphism)

Handbook of Combinatorial Designs (2007)



40 2-(v,k,λ) Designs of Small Order **II.1**

No	v	b	r	k	λ	Nd	Nr	Comments, Ref	Where?
196	*91*	*273*	*18*	*6*	*1*	≥ 4	-	*[534, 1607]*	*VI.16.18*
197	46	138	18	6	2	≥ 1	-	2#34*	VI.16.18
198	31	93	18	6	3	$\geq 10^{22}$	-	3#12 [1548]	
199	19	57	18	6	5	≥ 1535	-	D#232* [734]	II.7.46
200	16	48	18	6	6	$\geq 10^8$	-	3#13,2#35 [1548]	
201	28	72	18	7	4	≥ 392	?	2#36 [1224]	
202	64	144	18	8	2	≥ 121	≥ 121	2#37 [1225]	
203	145	290	18	9	1	?	-		
204	73	146	18	9	2	≥ 3500	-	2#38 [1239]	
205	49	98	18	9	3	≥ 1	-	[1628]	VI.16.30
206	37	74	18	9	4	≥ 852	-	2#39 [1224]	
207	25	50	18	9	6	≥ 79	-	2#40	
208	19	38	18	9	8	≥ 108	-	2#41,D#233 [734]	
209	55	99	18	10	3	?	-		
210	100	150	18	12	2	?	-		
211	34	51	18	12	6	≥ 2	-	R#218* [1496]	
212	85	102	18	15	3	?	-	R#217*	
213	136	153	18	16	2	?	-	R#216*	
214	289	306	18	17	1	≥ 1	≥ 1	R#215,AG(2,17)	
215	307	307	18	18	1	≥ 1	-	PG(2,17)	VI.18.73
216	154	154	18	18	2	?	-		
217	103	103	18	18	3	0	-	$\times 1$	
218	52	52	18	18	6	0	-	$\times 1$	
219	39	247	19	3	1	$\geq 10^{44}$	≥ 1626684	[1463, 562]	VI.16.12
220	20	95	19	4	3	≥ 10040	≥ 204	D#270 [1999, 623, 734]	VI.16.83
221	20	76	19	5	4	≥ 10067	≥ 14	D#271* [734, 1042]	VI.16.85
222	96	304	19	6	1	≥ 1	?	[1609]	II.3.32
223	153	323	19	9	1	?	?		
224	20	38	19	10	9	$\geq 10^{16}$	3	R#233,HD [1319]	
225	39	57	19	13	6	?	0	R#232*,$\times 3$	
226	96	114	19	16	3	?	0	R#231*,$\times 3$	
227	153	171	19	17	2	0	0	R#230*,$\times 2$	
228	324	342	19	18	1	?	?	R#229*,AG(2,18)	
229	343	343	19	19	1	?	-	PG(2,18)	
230	172	172	19	19	2	0	-	$\times 1$	
231	115	115	19	19	3	?	-		
232	58	58	19	19	6	0	-	$\times 1$	
233	39	39	19	19	9	$\geq 5.87 \cdot 10^{14}$	-	[1374]	V.1.28
234	21	140	20	3	2	$\geq 5 \cdot 10^{14}$	≥ 79	2#42,D#307 [1548]	
235	9	60	20	3	5	5862121434	203047732	5#2 [1707]	
236	6	40	20	3	8	13	1	4#4 [1174]	
237	61	305	20	4	1	≥ 18132	-	[1999, 587]	VI.16.61
238	31	155	20	4	2	≥ 43	-	[1999, 734]	VI.16.15
239	21	105	20	4	3	≥ 26320	-	D#308* [1999, 734]	VI.16.15
240	16	80	20	4	4	$\geq 6 \cdot 10^5$	$\geq 6 \cdot 10^5$	4#5 [1548]	
241	13	65	20	4	5	$\geq 10^3$	-	5#3 [1548]	
242	11	55	20	4	6	≥ 348	-	[734]	VI.16.15
243	81	324	20	5	1	≥ 1	-	[1999]	VI.16.16
244	41	164	20	5	2	≥ 6	-	2#45	
245	21	84	20	5	4	$\geq 10^9$	-	4#6,D#309 [1548]	II.7.46
246	17	68	20	5	5	≥ 7260	-	[514, 734]	VI.16.17
247	11	44	20	5	8	≥ 4394	-	4#7	
248	51	170	20	6	2	≥ 446	-	2#48* [2063]	VI.16.91
249	21	70	20	6	5	≥ 1	-	D#310 [1042]	VI.16.18
250	21	60	20	7	6	≥ 3810	≥ 1	2#49,D#311* [1]	

List goes up to $r = 41$ and has 1196 rows.

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215	307	307	18	18	1	≥ 1	-	PG(2,17)	VI.18.73
216	154	154	18	18	2	?	-		
217	103	103	18	18	3	0	-	$\times 1$	
218	52	52	18	18	6	0	-	$\times 1$	
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Data for a design

Need to list the points for each block.

In other words:

$k \times b$ matrix with entries from v symbols.

Other way:
$v \times b$ incidence matrix:
ij-entry is 1 if Point i belongs to Block j and 0 otherwise.

Symmetries can simplify this.

Automorphism group

A symmetry or automorphism of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation of the point set $\mathcal{P}$ that leaves the block set $\mathcal{B}$ invariant.

The set of all automorphisms forms a group under composition, denoted $\mathrm{Aut}(\mathcal{D})$.

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We say $G \leqslant \mathrm{Aut}(\mathcal{D})$ is transitive on X (say points or blocks) if for any two elements in X there is a permutation in G mapping one to the other.

Advantage of block-transitive designs: only need to know one block and generators for this group to describe the design.

Flag-transitive automorphism group

A flag is an incident point-block pair.

flag-transitive $\Rightarrow$ block-transitive and point-transitive

Flag-transitive automorphism group

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Block's Lemma: block-transitive $\Rightarrow$ point-transitive

Block-transitive automorphism group

Given a set $B \subset \mathcal{P}$ of size k and a permutation group G transitive on $\mathcal{P}$, when do we have that $(\mathcal{P}, B^G)$ is a 2-design?

Let $\mathcal{O}$ be an orbit of G on $\binom{\mathcal{P}}{2}$.

Double-counting $\{(\tilde{B}, \{\alpha, \beta\}) | \tilde{B} \in B^G, \alpha, \beta \in \tilde{B}, \{\alpha, \beta\} \in \mathcal{O}\}$ we get

$$b.|\binom{B}{2} \cap \mathcal{O}| = |\mathcal{O}|.\lambda_{\mathcal{O}}$$

Thus we have a 2-design iff $\frac{|\binom{B}{2} \cap \mathcal{O}|}{|\mathcal{O}|}$ is the same for each orbit $\mathcal{O}$.

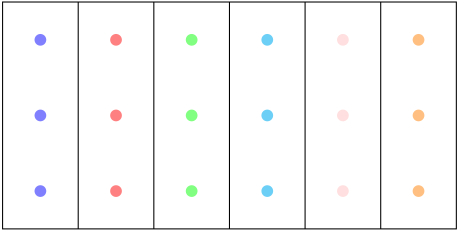
In particular this ratio is equal to $\frac{\lambda}{b} = \frac{k(k-1)}{v(v-1)}$.

Note we can actually drop the condition for one of the orbits.

Imprimitive group

We say a transitive permutation group G is imprimitive on X (say on points) if G preserves a non-trivial partition Σ of X.

$$\Sigma = \{C_1, C_2, \ldots, C_d\} \text{ with } |C_i| = c, \quad c > 1, d > 1$$



$|X| = 18$
$c = 3, d = 6$

For a set B of points, we set $x_i = |B \cap C_i|$.

Primitive group

Otherwise G is primitive on X: the only invariant partitions are trivial.

Higman-McLaughlin (1961): flag-transitive 2-$(v, k, 1)$ designs (linear spaces) are point-primitive.

This is not true for larger λ.

Imprimitive block-transitive designs

Delandtsheer-Doyen (1989): A point-imprimitive and block-transitive 2-design satisfies $v \leqslant (\binom{k}{2} - 1)^2$.

Suppose G is imprimitive and has two orbits on $\binom{\mathcal{P}}{2}$:
$\mathcal{O}_1$: point pair in same part
$\mathcal{O}_2$: point pair in different parts

We get one condition to get a block-transitive 2-design:

$$\frac{k(k-1)}{v(v-1)} = \frac{|\binom{B}{2} \cap \mathcal{O}_1|}{|\mathcal{O}_1|} = \frac{\Sigma \binom{x_i}{2}}{d\binom{c}{2}}$$

which simplifies to

$$\sum_{i=1}^{d} \binom{x_i}{2} = \binom{k}{2} \frac{c-1}{v-1}$$

Imprimitive flag-transitive designs

Stabiliser of B must be transitive on points of B so all non-zero x_i are equal.

Cameron-Praeger (1989): A point-imprimitive and flag-transitive 2-design satisfies $v \leqslant (k-2)^2$.

Davies (1987) showed that, for fixed λ, there are only finitely many flag-transitive, point-imprimitive 2-designs, by showing that the block-size k is bounded in terms of λ.

Imprimitive group and designs

With Cheryl Praeger: study of flag-transitive imprimitive 2-designs (2021,2023).

- $k \leqslant 2\lambda^2(\lambda - 1)$
- classification of all such designs for $\lambda \leqslant 4$ (11 of them).
- to describe the designs, we listed a block and a block-transitive group
- some admit multiple imprimitivity systems.

Example

$\mathcal{P} = \{(i, j) \mid 1 \leqslant i, j \leqslant 6\}$
$G = \{(g, g^{\sigma}) | g \in \mathrm{Sym}(6)\}$ where σ is an outer automorphism of $\mathrm{Sym}(6)$

	1	2	3	4	5	6
1	•	•	•	•	•	•
2	•	•	•	•	•	•
3	•	•	•	•	•	•
4	•	•	•	•	•	•
5	•	•	•	•	•	•
6	•	•	•	•	•	•

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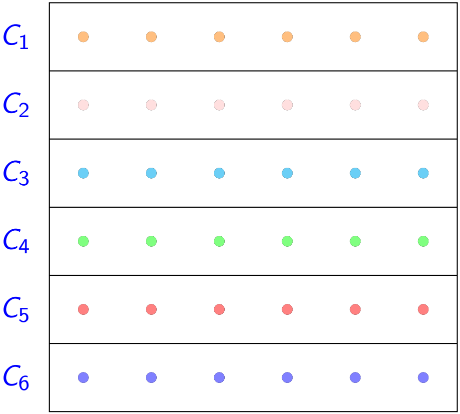

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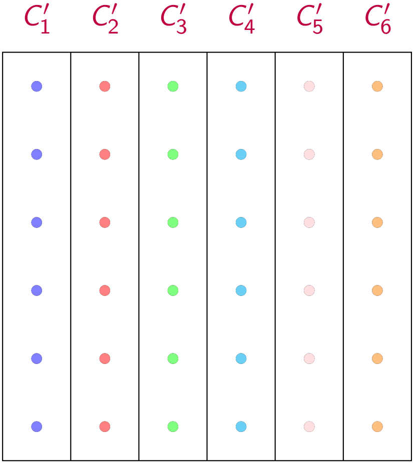

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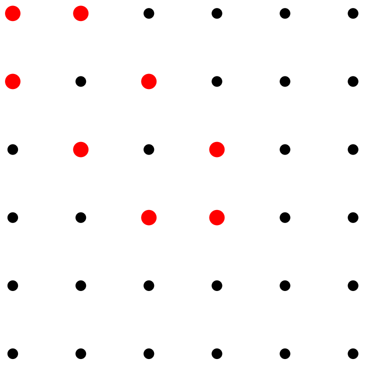


B=red points
$(\mathcal{P}, B^G)$ is a 2-$(36, 8, 4)$ design
G is full automorphism group
G is flag-transitive (even flag-regular)
Two imprimitivity systems forming a grid

Point pairs in same row forms an orbit on point-pairs

$$x_i = 2, 2, 2, 2, 0, 0 \longrightarrow \sum_{i=1}^{6} \binom{x_i}{2} = 4 = \binom{8}{2} \frac{5}{35}$$

Grid-imprimitive block-transitive 2-designs

Point-set is Cartesian product $\mathcal{P} = \mathcal{R} \times \mathcal{C}$ with $|\mathcal{R}| = m, |\mathcal{C}| = n$.

Group G preserving grid structure: $G \leqslant \mathrm{Sym}(\mathcal{R}) \times \mathrm{Sym}(\mathcal{C})$.

Two systems of imprimitivity:
$\Sigma = \{C_i\}$ where $C_i = \{(i, x)\}$ for $i = 1, \ldots, m$
$\Sigma' = \{C'_j\}$ where $C'_j = \{(x, j)\}$ for $j = 1, \ldots, n$

If $(\mathcal{P}, B^G)$ is a 2-design then so is $(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{R}) \times \mathrm{Sym}(\mathcal{C})})$.

Grid-imprimitive block-transitive 2-designs

Pick a k-subset B of $\mathcal{P}$.

Let $x_i = |B \cap C_i|$ and $y_j = |B \cap C'_j|$.

Theorem (Seyed Hassan Alavi, Ashraf Daneshkhah, AD and Cheryl E. Praeger 2023)

$(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{R})\times\mathrm{Sym}(\mathcal{C})})$ *is a 2-design if and only if*

1. $\displaystyle\sum_{i=1}^{m}\binom{x_i}{2} = \binom{k}{2}\frac{n-1}{v-1}$ *and*
2. $\displaystyle\sum_{j=1}^{n}\binom{y_j}{2} = \binom{k}{2}\frac{m-1}{v-1}.$

Very large λ.

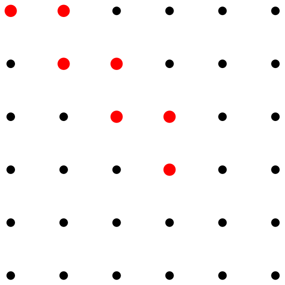
This design is by definition block-transitive.

It is also flag-transitive iff the stabiliser of B in $\mathrm{Sym}(\mathcal{R}) \times \mathrm{Sym}(\mathcal{C})$ is transitive on B.

Examples

$m = n$ even
$k = m + 1$

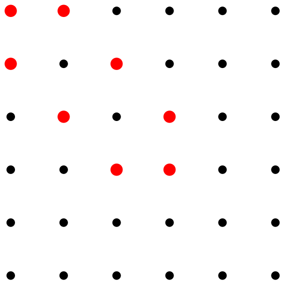
$(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{R})\times\mathrm{Sym}(\mathcal{C})})$ is a block-transitive 2-design



Examples

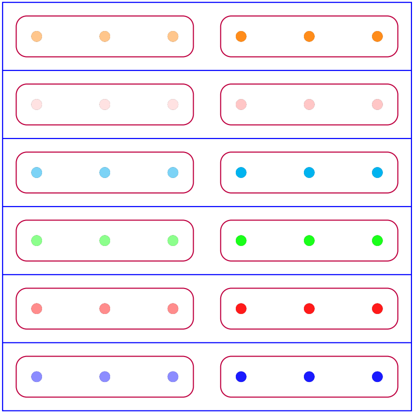
$m = n$ even
$k = m + 2$

$(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{R})\times\mathrm{Sym}(\mathcal{C})})$ is a flag-transitive 2-design



Chain of partitions

There is another way to have two imprimitivity systems.



chain of partitions
$\binom{\mathcal{P}}{1} = \mathcal{C}_0 \prec \mathcal{C}_1 \prec \mathcal{C}_2 \prec \mathcal{C}_3 = \{\mathcal{P}\}$.

Each $\mathcal{C}_i$-class is contained in a $\mathcal{C}_{i+1}$-class.

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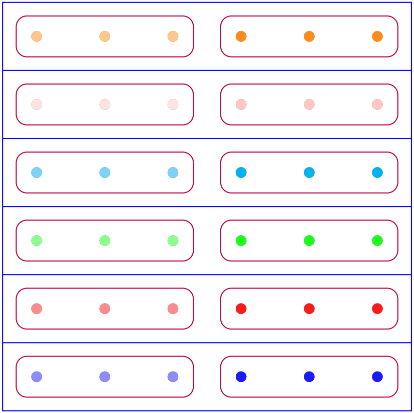

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Each $\mathcal{C}_i$-class is contained in a $\mathcal{C}_{i+1}$-class.

Each $\mathcal{C}_1$-class contains 3 $\mathcal{C}_0$-classes
Each $\mathcal{C}_2$-class contains 2 $\mathcal{C}_1$-classes
Each $\mathcal{C}_3$-class contains 6 $\mathcal{C}_2$-classes

Chain of partitions

We can have a longer chain of partitions.

$$\binom{\mathcal{P}}{1} = \mathcal{C}_0 \prec \mathcal{C}_1 \prec \cdots \prec \mathcal{C}_{s-1} \prec \mathcal{C}_s = \{\mathcal{P}\}.$$

Each $\mathcal{C}_i$-class is contained in a unique $\mathcal{C}_{i+1}$-class.

We set $e_i =$number of $\mathcal{C}_{i-1}$-classes contained in a $\mathcal{C}_i$-class, so that $v = |\mathcal{P}| = \prod_{i=1}^{s} e_i$.

Chain of partitions

The full group preserving this chain structure is an iterated wreath product $S_{e_1} \wr \ldots \wr S_{e_s}$

Group G preserving chain structure: $G \leqslant S_{e_1} \wr \ldots \wr S_{e_s}$. We say G acts s-chain-imprimitively.

If $(\mathcal{P}, B^G)$ is a 2-design then so is $(\mathcal{P}, B^{S_{e_1} \wr \ldots \wr S_{e_s}})$.

Chain-imprimitive block-transitive 2-designs

Pick a k-subset B of $\mathcal{P}$.
Let $x_C = |B \cap C|$ for each class C in one of the partitions.

Theorem (Carmen Amarra, AD, Cheryl E Praeger 2024)

$(\mathcal{P}, B^{S_{e_1} \wr \ldots \wr S_{e_s}})$ is a 2-design if and only if

1. $$\sum_{C \in \mathcal{C}_1} \binom{x_C}{2} = \binom{k}{2} \frac{e_1 - 1}{v - 1} \text{ and}$$
2. $$\sum_{C \in \mathcal{C}_i} x_C \left(x_{C^+} - x_C \right) = 2 \binom{k}{2} \frac{e_{i+1} - 1}{v - 1} \prod_{j \leqslant i} e_j \text{ for } i \in \{1, \ldots, s-1\}$$
   where C^+ =unique $\mathcal{C}_{i+1}$-class containing C.

In particular, $v - 1$ divides $\binom{k}{2} \cdot \gcd(e_1 - 1, \ldots, e_s - 1)$.

Here we have s conditions (but actually we can drop one).
The design is by definition block-transitive.

Examples

Consider a projective plane $\mathrm{PG}(2,q)$, where $q^2+q+1=\prod_{i=1}^{s} e_i$ for some pairwise coprime integers $e_1,\ldots,e_s$, with $e_i \geqslant 2$.

This is a 2-$(q^2+q+1, q+1, 1)$ design.

Then a subgroup G of automorphisms generated by a Singer cycle leaves invariant a chain of partitions and is block-transitive.

This gives us some parameters that work, which we were able to generalise.

Examples

Let $s \geqslant 2$. Let $p > 1$ a positive integer (not necessarily prime), $q := p^{2^{s-1}}$
Let

$$e_1 = p^2 + p + 1 \text{ and } e_i = p^{2^{i-1}} - p^{2^{i-2}} + 1 \text{ for } 2 \leqslant i \leqslant s.$$

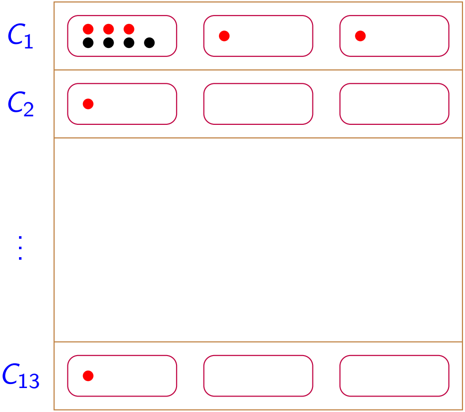
then $v = \prod_{i=1}^{s} e_i = q^2 + q + 1$. It is possible to find a set B of size $q + 1$ that satisfies the conditions so that $(\mathcal{P}, B^{S_{e_1} \wr \ldots \wr S_{e_s}})$ is a 2-design.

Theorem (Carmen Amarra, AD, Cheryl E Praeger 2024)

Given any integer $s \geqslant 2$, there exist infinitely many 2-designs which are block-transitive and s-chain-imprimitive for some automorphism group G.

Examples

Smallest example for $s = 3$: $p = 2, e_1 = 7, e_2 = 3, e_3 = 13, k = 17$



Not flag-transitive.

Chain-imprimitive flag-transitive 2-designs

Can a chain-imprimitive design be flag-transitive?
This requires the stabiliser of B in $S_{e_1} \wr \ldots \wr S_{e_s}$ to be transitive on B, and so $x_C \in \{0, y_i\}$ if C is a $\mathcal{C}_i$-class, where y_i depends only on i.

Theorem (Carmen Amarra, AD, Cheryl E Praeger 2024)

There exists a flag-transitive s-chain-imprimitive 2-(v, k, λ) *design* $(\mathcal{P}, B^{S_{e_1} \wr \ldots \wr S_{e_s}})$ *if and only if*

1. $v - 1 = (\prod e_i) - 1$ *divides* $(k-1)d$ *and*
2. *for each* $i \in \{1, \ldots, s-1\}$, $y_i := 1 + \frac{k-1}{v-1}\left(\left(\prod_{j \leqslant i} e_j\right) - 1\right)$ *is an integer dividing* $\frac{e_{i+1}-1}{d}\left(\prod_{j \leqslant i} e_j\right)$,

where $d = \gcd(e_1 - 1, \ldots, e_s - 1)$.

Purely numerical condition. Many solutions found computationally.
For $s = 3$, $e_1, e_2, e_3 \leqslant 50$: 57 examples, all with k quite large.

Chain-imprimitive flag-transitive 2-designs

Theorem (Carmen Amarra, AD, Cheryl E Praeger 2024)

There exists infinitely many flag-transitive s-chain-imprimitive 2-(v, k, λ) *designs for every* s.

For any value of $d \geqslant 2$, define

$$e_1 = d + 1, \quad e_i = d + \prod_{j \leqslant i-1} e_j, \text{ for } 2 \leqslant i \leqslant s, \text{ and } k = 1 + \frac{v-1}{d}.$$

These numbers satisfy the required conditions.

Smallest example, taking $d = 2$:
$e_1 = 3, e_2 = 5, e_3 = 17, v = 255, k = 128$
$y_1 = 2, y_2 = 8$

s-Grid-imprimitive block-transitive 2-designs

Point-set is Cartesian product $\mathcal{P} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \times \mathcal{E}_s$ with $e_i := |\mathcal{E}_i| > 1$.

Group G is s-grid imprimitive: $G \leqslant \mathrm{Sym}(\mathcal{E}_1) \times \mathrm{Sym}(\mathcal{E}_2) \times \ldots \times \mathrm{Sym}(\mathcal{E}_s)$.

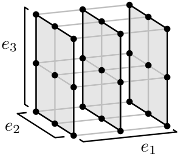
If $(\mathcal{P}, B^G)$ is a 2-design then so is $(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{E}_1) \times \mathrm{Sym}(\mathcal{E}_2) \times \ldots \times \mathrm{Sym}(\mathcal{E}_s)})$.

s-Grid-imprimitive block-transitive 2-designs

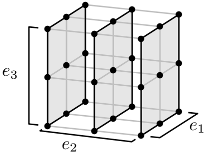
s systems of imprimitivity:

$\mathcal{C}_{\{i\}} = \{C_{\delta_i}\}$ where $C_{\delta_i} = \{(x_1, x_2, \ldots, x_s) | x_i = \delta_i\}$ for $\delta_i \in \mathcal{E}_i$

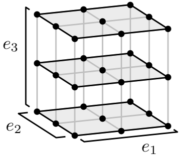
$\mathcal{C}_{\{1\}}$



$\mathcal{C}_{\{2\}}$

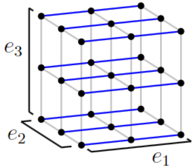


$\mathcal{C}_{\{3\}}$

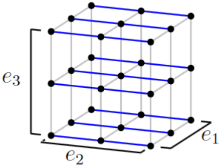


Intersections of these systems are also systems or imprimitivity.

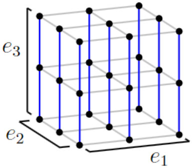
$\mathcal{C}_{\{2,3\}}$



$\mathcal{C}_{\{1,3\}}$



$\mathcal{C}_{\{1,2\}}$



Let $J \subseteq \{1, 2, \ldots, s\}$ and $\delta_J \in \mathcal{E}_J := \prod_{j \in J} \mathcal{E}_j$.

Define $\mathcal{C}_J = \{C_{\delta_J}\}$ where $C_{\delta_J} = \{(x_1, x_2, \ldots, x_s) | x_j = \delta_j$ for $j \in J\}$.

s-Grid-imprimitive block-transitive 2-designs

Pick a k-subset B of $\mathcal{P}$.
Let $x_C = |B \cap C|$ for each class C.

Theorem (Seyed Hassan Alavi, Carmen Amarra, Ashraf Daneshkhah, AD and Cheryl E. Praeger 2025)

$(\mathcal{P}, B^{\mathrm{Sym}(\mathcal{E}_1)\times\mathrm{Sym}(\mathcal{E}_2)\times\ldots\times\mathrm{Sym}(\mathcal{E}_s)})$ *is a 2-design if and only if for all* $\varnothing \neq J \subsetneq \{1, 2, \ldots, s\}$ *we have*

$$\sum_{C \in \mathcal{C}_J} x_C^2 = k + \frac{k(k-1)}{v-1}\left(\left(\prod_{i \notin J} e_i\right) - 1\right)$$

Here we have $2^s - 2$ conditions.

When $s = 2$ this is equivalent to the two conditions I showed earlier.

Examples

Again inspired by the projective plane $\mathrm{PG}(2, q)$, where $q^2 + q + 1 = \prod_{i=1}^{s} e_i$.

Let $s \geqslant 2$. Let $p > 1$ a positive integer (not necessarily prime), $q := p^{2^{s-1}}$. Let

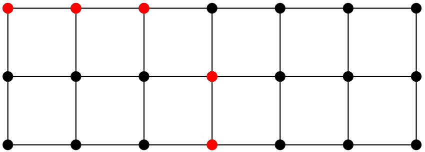
$$e_1 = p^2 + p + 1 \text{ and } e_i = p^{2^{i-1}} - p^{2^{i-2}} + 1 \text{ for } 2 \leqslant i \leqslant s.$$

then $v = \prod_{i=1}^{s} e_i = q^2 + q + 1$.

Again $k = q + 1$, so that $\frac{k(k-1)}{v-1} = 1$.

Examples

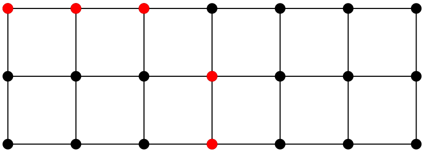
Smallest example for $s = 2$, take $p = 2$.
$e_1 = 7$, $e_2 = 3$
$k = 5$



B=red points

Examples

Smallest example for $s = 2$, take $p = 2$.
$e_1 = 7,\ e_2 = 3$
$k = 5$



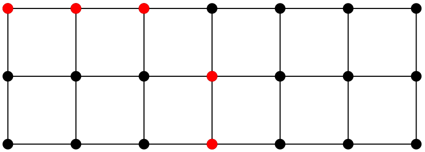
B=red points

$$x_i = 3, 1, 1 \longrightarrow \sum_{i=1}^{3} \binom{x_i}{2} = 3 = \binom{5}{2} \frac{6}{20}$$

$$y_i = 1, 1, 1, 2, 0, 0, 0 \longrightarrow \sum_{j=1}^{7} \binom{y_j}{2} = 1 = \binom{5}{2} \frac{2}{20}$$

Examples

Smallest example for $s = 2$, take $p = 2$.
$e_1 = 7,\ e_2 = 3$
$k = 5$



B=red points

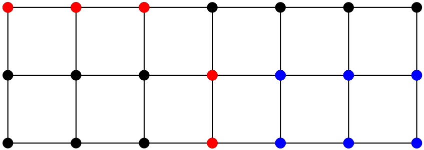
$$x_i = 3, 1, 1 \longrightarrow \sum_{i=1}^{3} x_i^2 = 11 = 5 + (e_1 - 1)$$

$$y_i = 1, 1, 1, 2, 0, 0, 0 \longrightarrow \sum_{j=1}^{7} y_j^2 = 7 = 5 + (e_2 - 1)$$

Examples

For $s = 3$, we managed to find a set B of size $q + 1$ that satisfies the conditions so that $(\mathcal{P}, B^{S_{e_1} \times S_{e_2} \times S_{e_s}})$ is a 2-design.
Much more complicated to describe B, the construction depends on p (mod 4).

$p = 2,\ e_1 = 7,\ e_2 = 3,\ e_3 = 13,\ k = 17$



red points in one $\mathcal{C}_{\{3\}}$-class
blue points represent pairs of points, all in different $\mathcal{C}_{\{3\}}$-classes

For $s = 4$, we only managed to find an example for $p = 2$.

Theorem (Seyed Hassan Alavi, Carmen Amarra, Ashraf Daneshkhah, AD and Cheryl E. Praeger 2025)

There exist infinitely many block-transitive, 3-grid-imprimitive 2-designs, and at least one block-transitive, 4-grid-imprimitive 2-design.

Open problems

Problem

Are there block-transitive s-grid-imprimitive 2-designs for any s or is there a bound on s?
If they exist, are there infinitely many?

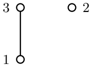
Problem

Are there flag-transitive s-grid-imprimitive 2-designs for $s \geqslant 3$?
If they exist, are there infinitely many?

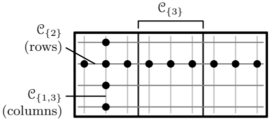
More complicated imprimitivity systems

There are other ways to arrange imprimitivity systems.

[R. A. Bailey, Cheryl E. Praeger, C. A. Rowley, and T. P. Speed, Generalized wreath products of permutation groups, 1983]

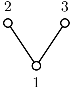


$(I, <)$



$G \leqslant (S_{e_1} \wr S_{e_3}) \times S_{e_2}$

More complicated imprimitivity systems



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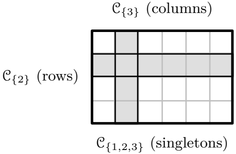
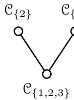
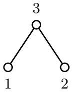




$(\mathfrak{C}, \prec)$

$$G \leqslant S_{e_1} \wr (S_{e_2} \times S_{e_3})$$



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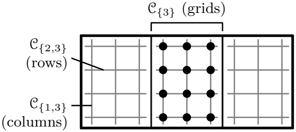

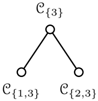

$(\mathfrak{C}, \prec)$

$$G \leqslant (S_{e_1} \times S_{e_2}) \wr S_{e_3}$$

More complicated imprimitivity systems

[Carmen Amarra, AD and Cheryl E. Praeger, work in progress.]

We determined general conditions for the incidence structure to be a 2-design.

We managed to find infinitely many examples in each of these 3 cases, again inspired by $\mathrm{PG}(2, q)$.

Thank you