Constructing highly regular expanders from hyperbolic Coxeter groups

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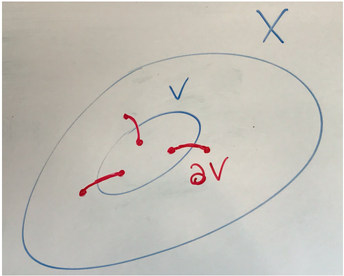
Expansion

A finite graph X is an ϵ-*expander*, if

$$h(X) = \min_{\emptyset \subsetneq V \subsetneq X} \frac{|\partial V|}{\min(|V|, |X \setminus V|)}$$

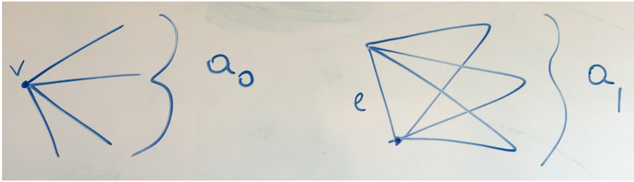
is at least ϵ. (∂V = edge-boundary of V).



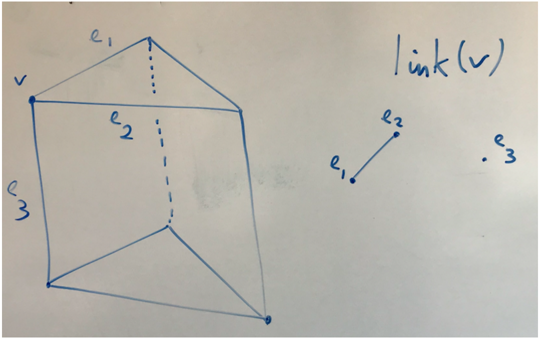
Regularity

X is $(a_0, \ldots, a_{n-1})$-regular of level n if

- X is a_0-regular, and
- $\forall v \in X$, the sphere of radius 1 around v is $(a_1, \ldots, a_{n-1})$-regular.



Connectivity



An (HR) graph with connected links will be called highly regular connected (HRC).

Chapman-Linial-Peled's question

- Chapman, Linial and Peled studied HRC-expander graphs of level 2 and ask whether such HRC-graphs of level 3 exist.
- We answer this question positively, also independently done by Friedgut and Iluz.
- Regularity and connectivity depend on the particular Coxeter diagrams.
- Expansion comes from superapproximation.

Polytopes and symmetry groups

Lemma

Let k be the largest integer for which $\mathcal{P}$ has a k-face which is a simplex, and suppose that $\mathrm{Aut}(\mathcal{P})$ acts transitively on the i-faces of $\mathcal{P}$ for $0 \leq i \leq n$. Then X (the 1-skeleton of $\mathcal{P}$) is an $(a_0, \ldots, a_{\min(k,n)})$-regular graph, where a_i is the number of simplicial $(i+1)$-faces containing a given i-face of $\mathcal{P}$. Moreover, X is $(a_0, \ldots, a_{\min(k,n)-1})$-regular connected.

Coxeter systems

Definition

$W = \langle\, S \mid (st)^{m_{st}} = 1 \forall s, t \in S \,\rangle$ where $m_{st} \in \{1, 2, \ldots, \infty\}$, $m_{st} = 1$ only if $s = t$.

1.  is $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$.
2.  is $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^4 = (su)^2 = 1 \rangle$.

Tits '61: To a string Coxeter system (W, S) one can associate a universal polytope $\mathcal{P}_W$ which is regular and for which $\mathrm{Aut}(\mathcal{P}_W) = W$.

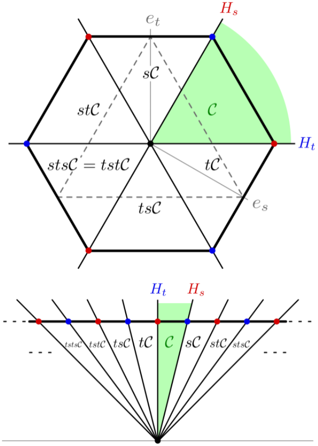
Geometric representation of a Coxeter group

Definition

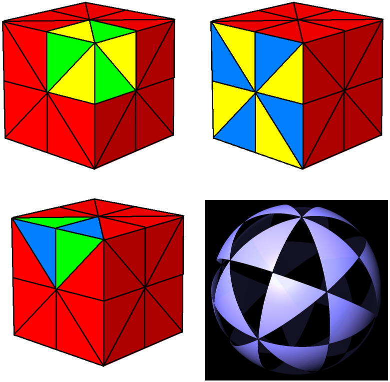
Set $B(e_s, e_t) = -\cos(\pi/m_{st})$. The *geometric representation of W* on $V = \mathbb{R}^S$ is defined by $s(v) = v - 2B(v, e_s)e_s$

- Tits: this representation is faithful.
- Image of W lies in orthogonal group O_B.
- The *signature of* (W, S) is defined to be the signature of B.

Two classic Coxeter complexes

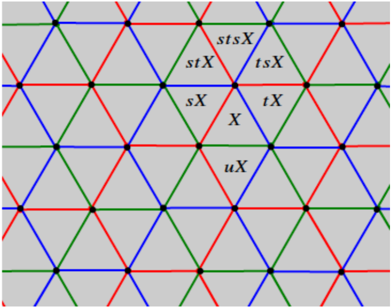


Another spherical example coming from the cube



- $EFE = FEF$, $VEVE = EVEV, VF = FV$

A Euclidean Coxeter complex



- $W := \langle s, t, u; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \rangle$

Wythoffian polytopes

- They form a class of uniform polytopes, i.e. $\mathrm{Aut}(\mathcal{P})$ acts transitively on vertices, and faces are inductively uniform.
- Not all uniform polytopes are Wythoffian, first counterexample: the grand antiprism (Conway and Guy 1965).
- Kaleidoscopic construction, for example octahedron, cuboctahedron and cube.

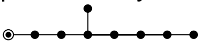
Main result

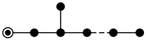
Theorem

Let (W, S) be a Coxeter system, M a subset of S and $\mathcal{P}_{W,M}$ the associated Wythoffian polytope. Suppose (W, S) is indefinite, $\mathcal{P}_{W,M}$ has finite vertex links, and the 1-skeleton X of $\mathcal{P}_{W,M}$ is $(a_0, \ldots, a_n)$-regular. Then there exists an infinite collection of finite quotients of X by normal subgroups of W, which form a family of $(a_0, \ldots, a_n)$-regular expander graphs.

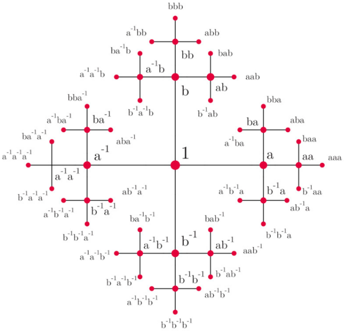
Illustrating the main theorem

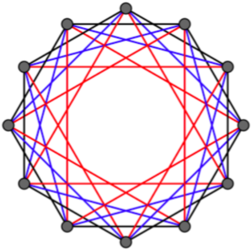
- $(120, 12, 5, 2)$-regular connected expander graphs, quotients of the 1-skeleton of the hyperbolic tessellation with diagram


- $(2160, 64, 21, 10)$-regular connected expander graphs from Wythoffian polytope with diagram 

- For each $m \geq 10$, there is a family of $(2^{m-2}, \frac{(m-1)(m-2)}{2}, 2(m-3))$-regular connected expanders as quotients of the polytope of type E_m with diagram


Groups and their Cayley graphs





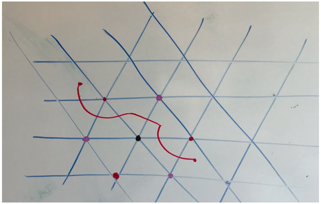
$\mathrm{Cay}(\mathbb{Z}_{12}; \{2,3,4\}^{\star})$

Quasi-isometry

Let X and Y be metric spaces. A map $f : X \to Y$ is a quasi-isometry if there exist constants $A \geq 1, B \geq 0, C \geq 0$ such that

$$\frac{1}{A} d(x, x') - B \leq d(f(x), f(x')) \leq A d(x, x') + B$$

$$\forall y \in Y : \exists x \in X : d(f(x), y) \leq C$$



From $\mathrm{Cay}(W, S)$ to $\mathcal{P}_{W,M}$

Lemma

Let (W, S) be Coxeter system and M a subset of S. The 1-skeleton X of the associated Wythoffian polytope $\mathcal{P}_{W,M}$ and the Cayley graph $\mathrm{Cay}(W, S)$ are quasi-isometric if and only if $\mathcal{P}_{W,M}$ has finite vertex links. In this case, the natural W-equivariant surjection $f : \mathrm{Cay}(W, S) \to X$ that sends a chamber to the unique vertex of $\mathcal{P}_{W,M}$ it contains is a nonexpansive quasi-isometry.

Comparing quotients

- Assume $\mathcal{P}_{W,M}$ has finite vertex links.
- Let $\pi_N : W \to W/N$ be the quotient map.
- $\mathrm{Cay}(W, S)/N \cong \mathrm{Cay}(\pi_N(W), \pi_N(S))$.
- There exists quasi-isometries f_N with the same constants as f, in particular **independent** of N.

$$\begin{tikzcd}
\mathrm{Cay}(W, S) \arrow[r, "f"] \arrow[d, "\pi_N"'] & X \arrow[d] \\
\mathrm{Cay}(\pi_N(W), \pi_N(S)) \arrow[r, "f_N"'] & X/N
\end{tikzcd}$$

Comparing regularity I

- Goal: X/N retains the regularity of X.
- Sufficient condition: $X \to X/N$ is injective on the neighbourhood of any vertex of X and creates no new triangles.
- Action of N on X should have minimal displacement (md) at least 4, thus action of N on $\mathrm{Cay}(W, S)$ had md at least $5D + 4$, i.e. $l(n) \geq 5D + 4$, $\forall n \neq 1 \in N$.
- The elements in W whose lengths are less than $5D + 4$ form a finite set T.

Comparing regularity II

- W is a finitely generated linear group, hence residually finite (Malcev 1940).
- Let $\{N_m\}_{m\in I}$ be finite-index normal subgroups of W closed under intersection with $\bigcap_{m\in I} N_m = \{1\}$, and let $I' = \{m \in I \mid T \cap N_m = \{1\}\}$, so that $\bigcap_{m\in I'} N_m = \{1\}$. For $m \in I'$ the graph X/N_m has the same regularity as X.
- If W is infinite then indices of the N_m are unbounded (f.g. groups only have finitely many subgroups of a given finite index).

Comparing expansion I

Proposition

Let $D \geq 1$ and let $f : Y \to Z$ be a D-quasi-isometry between two finite connected graphs Y and Z. Then there exist constants $c, c' > 0$ depending only on the quasi-isometry constants of f (or equivalently, on D) and on the maximum degrees of Y and Z, such that if $h(Y) \geq \epsilon$, then $h(Z) \geq \min(c\epsilon, c')$.

Comparing expansion II

Corollary

Let $\{Y_m\}_{m \in J}$ and $\{Z_m\}_{m \in J}$ be two families of graphs of bounded maximum degree, indexed by a set J. Suppose that there is a D-quasi-isometry $f_m : Y_m \to Z_m$ for every $m \in J$. Then $\{Y_m\}_{m \in J}$ is a family of expanders if and only if $\{Z_m\}_{m \in J}$ is.

Why indefinite Coxeter groups?

- Since S is assumed to be finite, W is a discrete subgroup of $\mathrm{O}_B(\mathbb{R})$.
- So if (W, S) is semidefinite (resp. definite), then W is virtually abelian (resp. finite).
- Virtually abelian groups are amenable, so there is no hope for expansion phenomena if W is semidefinite.

Proof of the main result

- $\{\mathrm{Cay}(\pi_m(W), \pi_m(S))\}_m$ forms a family of expanders for an appropriate family of m's (superapproximation).
- $f_m : \mathrm{Cay}(\pi_m(W), \pi_m(S)) \to X/N_m$ with constants depending only on (W, S).
- $\{X/N_m\}_{m \in I}$ form a family of expanders.
- Let $I' = \{m \in I \mid X/N_m \text{ same regularity as } X\}$.
- The graphs $\{X/N_m\}_{m \in I'}$ are $(a_0, \ldots, a_n)$-regular, and form an infinite family of expanders.

The order-5-4-simplex-honeycomb

The automorphism group of $\mathcal{P}$ is the Coxeter group W with diagram $\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\overset{5}{-}\bullet$, (H_5).

Let $\varphi = \frac{1+\sqrt{5}}{2} \in \mathbb{R}$ and let $K = \mathbb{Q}(\varphi)$. Then the matrix of B is

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\varphi \\ 0 & 0 & 0 & -\varphi & 2 \end{pmatrix}$$

w.r.t. the canonical basis. B is equivalent over K to $B' = \langle 1, 1, 1, 1, -\varphi \rangle$. Hence $\mathrm{O}_B \cong \mathrm{O}_{B'}$ as algebraic K-groups.

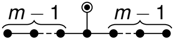
Two-sheeted hyperbola

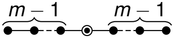
- $\{v \in \mathbb{R}^5 \mid B(v,v) = -1\}$ is preserved by O_B. Both sheets $\mathcal{H}$ and $\mathcal{H}^-$ are Minkowski models for hyperbolic 4-space and preserved by W.
- $\mathrm{Isom}(\mathcal{H}) = \mathrm{O}_B^+(\mathbb{R}) = \{g \in \mathrm{O}_B(\mathbb{R}) \mid g\mathcal{H} = \mathcal{H}\} \xrightarrow{\sim} \mathcal{P}\mathcal{O}_B(\mathbb{R})$.
- The images of $\{s_0, \ldots, s_4\}$ of W lie in $\mathrm{O}_B^+(\mathcal{O}_K)$. The hyperplanes they generated tessellate $\mathcal{H}$ by compact 4-simplices, and form a geometric representation of the Coxeter complex of W.

The geometry of $\mathcal{P}$

- The link L of a vertex of $\mathcal{P}$ is a hexacosichoron (600-cell) and the link of an edge of $\mathcal{P}$ is an icosahedron.
- Hence W is a cocompact lattice in $\mathrm{O}_B(\mathbb{R})$, and by Borel density W is Zariski-dense in $\mathrm{O}_B(\mathcal{O}_K)$.
- W has finite index in $\mathrm{O}_B(\mathcal{O}_K)$ ($\mathrm{O}_B(\mathcal{O}_K)$ is a discrete subgroup of $\mathrm{O}_B(\mathbb{R})$ containing W).
- String Coxeter diagram, and hence $(120, 12, 5, 2)$-regular connected expanders.

Arbitrarily high regularity levels

For any $m \geq 5$ let $\mathcal{P}_m$ be 

The 1-skeleton X_m of $\mathcal{P}_m$ is a $(\binom{2m}{m}, m^2, 2(m-1), m-2, m-3, \ldots, 1)$-regular graph. The link of any vertex in $\mathcal{P}_m$ is an m-rectified $(2m-1)$-simplex, with diagram 

, and the 1-skeleton of this link is the Johnson graph $J(2m, m)$.

The work of Friedgut and Iluz

- “Hyper-regular graphs and high dimensional expanders."
- They observed H_5 leads $(120, 12, 5, 2)$-regular graphs, and Friedgut had presented this at MFO in April 2019, but with no mention of the expansion of those graphs.
- They also have a method to show that $\mathrm{HRC}_\infty(n)$ and even $\mathrm{HRC}_{\mathrm{exp}}(n)$ are infinite.

Two open problems

Problem A:

$$
\begin{array}{ccccc}
\mathrm{HRC}_{\mathsf{exp}}(n) & \hookrightarrow & \mathrm{HRC}_{\infty}(n) & \hookrightarrow & \mathrm{HRC}(n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{HR}_{\mathsf{exp}}(n) & \hookrightarrow & \mathrm{HR}_{\infty}(n) & \hookrightarrow & \mathrm{HR}(n)
\end{array}
$$

Are any of these inclusions strict for $n > 1$?

Problem B: For $n > 1$ describe the above six sets as subsets of $\mathbb{N}^n$.

Superapproximation

Fix $N_0, q_0 \in \mathbb{N}_0$. For m coprime to q_0, let $\pi_m = \mathrm{GL}_{N_0}(\mathbb{Z}[1/q_0]) \to \mathrm{GL}_{N_0}(\mathbb{Z}/m\mathbb{Z})$.

Theorem (Salehi-Golsefidy)

Let $\Gamma = \langle S \rangle$ where $S = S^{-1} \subset \mathrm{GL}_{N_0}(\mathbb{Z}[1/q_0])$. Suppose that Γ is infinite. Fix $M_0 \in \mathbb{N}$. The family of Cayley graphs $\{\mathrm{Cay}(\pi_m(\Gamma), \pi_m(S))\}_m$, as m runs through either $\{p^n \mid n \in \mathbb{N}, p \; prime, p \nmid q_0\}$ or $\{m \in \mathbb{N} \mid \gcd(m, q_0) = 1, p^{M_0+1} \nmid m \; for \; p \; prime\}$, is a family of expanders if and only if the connected component G° of the Zariski-closure G of Γ in GL_{N_0} is perfect.

Weil's restriction of scalars

- The entries of the matrix of $2B$ in the canonical basis of V are algebraic integers, and so there exists a number field K, with ring of integers $\mathcal{O}_K$, over which O_B can be defined such that $W \subset \mathrm{O}_B(\mathcal{O}_K)$.
- The restriction of scalars $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{O}_B)$ is a linear algebraic $\mathbb{Q}$-group, and as such can be embedded over $\mathbb{Q}$ in GL_{N_0} for some N_0.
- Let q_0 be the lcd of the entries of the image of S. Then $W \subset \mathrm{GL}_{N_0}(\mathbb{Z}[1/q_0])$. The Zariski-closure of W in GL_{N_0} is the image of $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{O}_B^{1\circ})$, which is perfect since $\mathrm{O}_B^{1\circ}$ is perfect.

Benoist-de la Harpe

Theorem (Benoist-de la Harpe)

Let (W, S) be an indefinite and irreducible Coxeter system. Then the Zariski-closure of W in O_B is precisely the kernel O_B^1 of the restriction map $\mathrm{O}_B \to \mathrm{GL}_{\mathrm{rad}(B)} : g \mapsto g_{|_{\mathrm{rad}(B)}}$. In particular, if B is non-degenerate, then W is Zariski-dense in O_B.

Connected component is perfect

The connected component $\mathrm{O}_B^{1\circ}$ of the Zariski-closure of the indefinite Coxeter group W is perfect. Indeed, let $V' = V/\operatorname{rad}(B)$, then $\mathrm{O}_B^{1\circ} \cong \mathrm{SO}_{B'} \ltimes V'^{\dim \operatorname{rad}(B)}$, with the latter being a perfect group because $\mathrm{SO}_{B'}$ is simple and V' is an irreducible $\mathrm{SO}_{B'}$-module.