On colouring problems for projective planes

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I'll NOT talk about ...

chromatic number of graphs associated to geometries, although there are many nice results about them. Even for large independent subsets in such graphs (EKR problems).

I can mention almost everyone who is attending the conference.

Results on q-analogues GODSIL, MEAGHER,BLOKHUIS, BROUWER, CHOWDHURY, FRANKL, MUSSCHE, PATKÓS, SZT,PEPE, STORME, VANHOVE, DE BEULE, METSCH, D'HAESELEER, HEERING,....

Probably I forgot several names, I apologize for it. It is a very nice topic where one can combine combinatorial, geometric, and algebraic, (in particular, eigenvalue) techniques.

Aim + Coauthors

In this talk we focus on (hypergraph) colouring problems related to projective planes., in particular about 2-colourings. The main new results about a conjecture of Erdős.

We also try to illustrate the difference between Galois planes and arbitrary ones.

This talk is based on joint work with AART BLOKHUIS, ÁDÁM MARKÓ, and ZSUZSA WEINER in the new part about Erdős colouring (that is 2-colourings of proj. planes with many balanced lines).

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Notation, chromatic number of proj. planes

The Galois plane over the field with q elements ($\mathrm{GF}(q)$): $\mathrm{PG}(2, q)$.
An arbitrary proj. plane of order q: Π_q.
The chromatic number of Π_q is 2 for $q > 2$, and 3 for $q = 2$.
A 4-elt. subset either contains a line, or it is the complement of a line.
For $q > 2$ the colour classe are a non-trivial blocking set and its complement.

How to make colourings of planes interesting?

Consider more general structures (e.g. uniform hypergraphs)
Impose extra conditions on colour class, for example:

1. colour classes are arcs (at most 2 points on a line can have the same colour)
2. colour classes are blocking sets (each line contains a point of each colour class)

In other words, modify the notion of chromatic number, e.g. by forbidding rainbow lines (instead of monochromatic ones), sometimes changing minimum/maximum no. of colours.

Property B

ERDŐS-HAJNAL if the number m of edges in an r-uniform hypergraph satisfies $m \leq 2^{r-1}$, then it is 2-colourable

ERDŐS there exists an r-uniform hypergraph with $m \leq cr^2 2^r$ edges that is not 2-colourable

Notation: $m(r)$ is the **minimum** number of edges in a non-2-colourable r-uniform hypergraph

BECK: $m(r) > c2^r r^{1/3}$,
RADHAKRISHNAN-SRINIVASAN: $m(r) > c2^r (r/\ln r)^{1/2}$.
Actual value $c = 0.7$.
Greedy colourings: PLUHÁR.

Variants of the previous results

ERDŐS-LOVÁSZ (using the famous Lovász Local Lemma)
If every edge meets $\leq 2^{r-3}$ other edges $\Longrightarrow \mathcal{H}$ is 2-colourable

If $\mathcal{H}$ is linear (two edges meet in at most one point) and

1. if each point has degree $\leq 2^{r-3}/r \Longrightarrow \mathcal{H}$ is 2-colourable
2. $n \leq 2^{r-4} \Longrightarrow \mathcal{H}$ is 2-colourable
3. $m \leq 4^{r-4}/r^3 \Longrightarrow \mathcal{H}$ is 2-colourable

RADHAKRISHNAN-SRINIVASAN: if every edge meets at most $0.172^r(r/\ln r)^{1/2}$ other edges $\Longrightarrow \mathcal{H}$ is 2-colourable.

Csima-Füredi

CSIMA, FÜREDI for the case when colour classes are arcs: in Π_q one needs at least $q+1$ colours, and it is sharp for Galos planes; $\mathrm{PG}(2,q)$ can be partitioned into q conics and a point.
For the case when colour classes are blocking sets ("colourful colourings")we want many colours; they refer to an unpublished result by
ERDŐS, T. SÓS saying that $\chi \geq q/(2\log q)$
and there are other nice results.
Geometrically we want to partition the points in disjoint blocking sets: (e.g. partition into Baer subplanes). CSIMA, FÜREDI show that it is optimal. From geometry: BEUTELSPACHER, EUGENI, BARÁT, MARCUGINI, PAMBIANCO, SzT.
J. Csima, Z. Füredi, Colouring finite incidence structures, *Graphs and Combinatorics* **2** (1986), 339-346.

Alon-Füredi

ALEX ROSA asked for the min. no. of colours, so that different lines have different colour distributions.

ALON, FÜREDI: the answer is between 5 and 8.

N. Alon, Z. Füredi, Legitimate colorings of projective planes, *Graphs and Combinatorics* **5** (1989), 95-106.

Upper chromatic number

Colour the vertices of a hypergraph $\mathcal{H}$.

A hyperedge is *rainbow*, if its vertices have pairwise distinct colors.

The upper chromatic number of $\mathcal{H}$, $\bar{\chi}(\mathcal{H})$: the maximum number of colors that can be used without creating a rainbow hyperedge (V. Voloshin).

For graphs, it gives the number of connected components.
Note that in case of the ordinary chromatic number we wish to avoid monochromatic hyperedges.
Exercise: The upper chromatic no. of the Fano-plane is 3.

Trivial coloring

$v := q^2 + q + 1$, the number of points in Π_q.

$\tau_2 :=$ the size of the smallest double blocking set in Π_q.

Then $\bar{\chi}(\Pi_q) \geq v - \tau_2 + 1$.

We call this a *trivial coloring*. General aim: show that the above bound is sharp.
In general, the union of colour classes of size at least two is a double blocking set.

Projective planes

Theorem (Bacsó, Tuza, 2007)

As $q \to \infty$,

- $\bar{\chi}(\Pi_q) \leq v - (2q + \sqrt{q}/2) + o(\sqrt{q})$;
- *for q square,* $\bar{\chi}(\mathrm{PG}(2,q)) \geq v - (2q + 2\sqrt{q} + 1) = v - \tau_2 + 1$;
- $\bar{\chi}(\mathrm{PG}(2,q)) \leq v - (2q + \sqrt{q}) + o(\sqrt{q})$;
- *for q non-square,* $\bar{\chi}(\mathrm{PG}(2,q)) \leq v - (2q + Cq^{2/3}) + o(\sqrt{q})$.

Theorem (Bacsó, Héger, SzT)

Let Π_q be an arbitrary projective plane of order $q \geq 4$, and let $\tau_2(\Pi_q) = 2(q+1) + c(\Pi_q)$. Then

$$\bar{\chi}(\Pi_q) < q^2 - q - \frac{2c(\Pi_q)}{3} + 4q^{2/3}.$$

Theorem (Bacsó, Tuza, 2007)

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- $\bar{\chi}(\Pi_q) \leq v - (2q + \sqrt{q}/2) + o(\sqrt{q})$;
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What do we know about double blocking sets?

In general, for any Π_q, we have $\tau_2 \leq 3q$, by taking three non-concurrent lines. For $\mathrm{PG}(2,q)$, we have better examples, except when q is a prime.

The smaller examples come from taking two disjoint blocking sets (e.g. two disjoint Baer-subplanes).

For q prime, there are double blocking sets in $\mathrm{PG}(2,q)$ with $3q-1$ points:

Csajbók, Héger, 2019: $q = 13, 19, 31, 37, 43$. (The construction also works for $q = 16, 25, 27$.)

The first examples for $q = 13$ were constructed by Braun, Kohnert, Wassermann, 2005.

Note that for q prime we only know $\tau_2(\mathrm{PG}(2,q)) \geq 2q + (q+5)/2$ (Ball).

Results for small double blocking sets in $\mathrm{PG}(2, q)$

A double blocking set is small, if it has at most $2q + (q + 1)/2$ points.
The proof uses that a double blocking set of size at most $3q$ contains a unique double blocking set. (see Harrach, SzT).
We also rely on results on small blocking sets, by Blokhuis, Storme, SzT, and by Blokhuis, Lovász, Storme, SzT. In particular, they showed that a small minimal double blocking set meets every line in 2 modulo p points, where q is a power of p. There are further improvements for blocking sets by Sziklai.
Regarding the existence of disjoint blocking sets, see Davydov, Giulietti, Marcugini, Pambianco, Polverino, Storme, and also the thesis by Van de Voorde.

Improvement for projective planes

Theorem (Bacsó, Héger, SzT)

Let $v = q^2 + q + 1$. Suppose that $\tau_2(\mathrm{PG}(2,q)) \leq c_0 q - 8$, $c_0 < 8/3$, and let $q \geq \max\{(6c_0 - 11)/(8 - 3c_0), 15\}$. Then

$$\bar{\chi}(\mathrm{PG}(2,q)) < v - \tau_2 + \frac{c_0}{3 - c_0}.$$

In particular, $\bar{\chi}(\mathrm{PG}(2,q)) \leq v - \tau_2 + 7$.

The above thm is (or can be) relevant for q prime.

Theorem (Bacsó, Héger, SzT)

Let $q = p^h$, p prime. Suppose that either $q > 256$ is a square, or $h \geq 3$ odd and $p \geq 29$. Then $\bar{\chi}(\mathrm{PG}(2,q)) = v - \tau_2 + 1$, and equality is reached only by trivial colorings.

G. Bacsó, Zs. Tuza, Upper chromatic number of projective planes, *Journal of Combin. Designs* **7** (2007), 39-53.
G. Bacsó, T. Héger, T. Szőnyi, The 2-blocking number and the upper chromatic number of PG$(2, q)$, *Journal of Combin. Designs* **21** (2013), 585-602.

Balanced upper chromatic number

Definition (balanced upper chromatic number)

The **balanced upper chromatic number** of a hypergraph $\mathcal{H}$ is the largest positive integer N for which $\mathcal{H}$ admits a rainbow-free strict N-coloring so that the color classes have almost the same size. Denote this number by $\overline{\chi}_b(\mathcal{H})$. It was introduced by Araujo-Pardo, Kiss, Montejano.

So, we would like to avoid the situation for trivial colorings: one large color class, and several one-element classes (but we wish to have as many colors as possible).

Known results

Theorem (Araujo-Pardo,Kiss,Montejano, 2015)

All balanced rainbow-free colorings of any projective plane of order q satisfies that each color class contains at least three points. Thus

$$\overline{\chi}_b(\Pi_q) \leq \frac{q^2+q+1}{3}.$$

Theorem (Araujo-Pardo, Kiss, Montejano, 2015)

Every cyclic projective plane of order q has a balanced rainbow-free coloring with at least $\frac{q^2+q+1}{6}$ color classes. Thus

$$\overline{\chi}_b(\Pi_q) \geq \frac{q^2+q+1}{6}.$$

Moreover, if $q \equiv 1 \pmod 3$, then $\overline{\chi}_b(\Pi_q) = \frac{q^2+q+1}{3}$.

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Moreover, if $q \equiv 1 \pmod 3$, then $\overline{\chi}_b(\Pi_q) = \frac{q^2+q+1}{3}$.

Theorem (Blázsik, Blokhuis, Miklavič, ZL Nagy, SzT, 2021)

Let $\mathrm{PG}(2, q)$ be the desarguesian projective plane of order q. Then

$$\overline{\chi}_b(\mathrm{PG}(2, q)) = \left\lfloor \frac{q^2 + q + 1}{3} \right\rfloor .$$

What type of results do we use?

It was already observed by Araujo-Pardo, Kiss and Montejano that it is enough to have a difference set containing $0, 1, 3$ and have color classes consisting of 3 consecutive integers (with one class possibly consisting of 4 consecutive integers).
A (projective) difference set comes from a primitive polynomial $p(x)$ of degree three and a subspace of dimension 2. If we take the subspace generated by 1 and x, and choose a polynomial of the form $p(x) = x^{-}bx - c$, then the subspace contains $1 = x^0$, $x = x^1$, and $x^3 = bx + c$, hence the difference set will contain $0, 1, 3$.
Luckily, S. D. Cohen proved that for $q \neq 4$,there is such a primitive polynomial (and $q = 4$ was done earlier).

A result for general (non-desarguesian) planes

Theorem (Blázsik, Blokhuis, Miklavič, ZL Nagy, SzT, 2021)

Let Π_q be an arbitrary projective plane of order $q > 133$. Then, its balanced upper chromatic number can be bounded from below as follows

$$\overline{\chi}_b(\Pi_q) \geq \frac{q^2 + q - 16}{10}.$$

In other words, the result says that we could show the existence of a coloring with colour classes of size 10 and 11, and without rainbow lines. The proof uses the probabilistic method (random coloring / recoloring and careful estimates for the number of rainbow lines) and also HAEMERS' incidence bound.

For $q \leq 133$, one can find, using a computer, a colouring with colour classes of size 11 and 12, without rainbow lines.

The papers on balanced upper chr. no

G. Araujo-Pardo, Gy. Kiss, A. Montejano, On the balanced upper chromatic number of cyclic projective planes and projective spaces, *Discrete Math.* **338** (2015), 2562-2571.

Z. L. Blázsik, A. Blokhuis, Š. Miklavič, Z. L. Nagy, T. Sz onyi, On the balanced upper chromatic number of finite projective planes, *Discrete Math.* **344** (2021), 112266.

Discrepancy

2-colouring: a function $f : V \to \{-1, +1\}$

discrepancy of an edge E: is $|\sum_{x \in E} f(x)| = \mathrm{disc}(E)$.
discrepancy of f: $\max\{\mathrm{disc}(E) : E \in \mathcal{H}\}$.
discrepancy of $\mathcal{H}$: $\min_f \max\{\mathrm{disc}(E) : E \in \mathcal{H}\}$.

ERDŐS: $\mathrm{disc}(\mathcal{H}) \leq \sqrt{2n \log(2m)}$, where $|V| = n$, and the hypergraph has m edges

SPENCER: For $m = n$, $\mathrm{disc}(\mathcal{H}) < 6\sqrt{n}$.
$\sqrt{n}$ is the right order of magnitude: Hadamard-designs

Discrepancy of projective planes = motivation for Erdős' colouring problem

Let $\mathcal{P}$ denote the set of red points, $|\mathcal{P}| > (q^2+q)/2$. List the lines L_i and let $r_i = |L_i \cap \mathcal{P}|$. Then

$$\sum r_i = |\mathcal{P}|(q+1), \text{and}$$

$$\sum r_i(r_i - 1) = |\mathcal{P}|(|\mathcal{P}| - 1).$$

Now $\mathrm{disc}(L_i) = |q+1-2r_i|$. and from the standard eqns we can compute $\sum \mathrm{disc}(L_i)^2$. This gives

$$\mathrm{disc}(\Pi_q) > \sqrt{q}.$$

On the other hand, SPENCER proved that $\mathrm{disc}(\Pi_q) \leq c'\sqrt{q}$.
J. Spencer, Coloring the projective plane, *Discrete Math.* **73** (1988-89), 213-220.

Erdős' colouring problem

Colour the pts of Π_q with two colours. A line is balanced it it has the same no. of red/blue pts (q odd). Unbalanced = not balanced

What is the max. no. of balanced lines?

(From discrepancy lower bound we see that there are unbalanced lines.)

We aim at a lower bound for the no. of unbalanced lines.

RÉDEI's direction problem: take a set $U \subset \mathrm{AG}(2, q)$ of size q.
Then there are determined directions (D) and non-determined directions (N).
From the points of N the set looks the same: one point on each line
GHIDELLI's generalization: Let U have nq points. The analogue of N is the set of infinite points, so the each line contains n points of U. Remaining dir.: analogue of D, called special directions
Ghidelli obtained nice results about the size of special dirs.
1 special dir.: parallel lines, 2 specials dirs: not possible, 3 special dirs? (next slide)
L. Ghidelli, On rich and poor directions ..., *Discr. Math.* **343** (2020), 111811.

The Kiss-Somlai example

Theorem (Kiss, Somlai)

Let $p > 2$ be a prime. Then the set

$$S = \{(x, y) \in \mathbb{F}_p^2 \ : y < x\}$$

has exactly three special directions, namely (0), (1) and (∞). Here the elements are integers between 0 and $p - 1$ and $<$ is the usual one.
Let M be a point set in $\mathrm{AG}(2, p)$. Let (d) be a non-special direction. If M admits at most three special directions, then M is either a union of r parallel lines or, up to an affine transformation, it is equivalent to S or its complement in the previous theorem.

G. Kiss, G. Somlai, Special directions on the finite affine plane, *Designs, Codes and Cryptography* **92** (2024), 2587-2597.

Extension to multisets

ADRIAENSEN, WEINER: instead of points below $Y = X$, the studied points below the parabola $Y = X^2$. They showed that the infinite points look the same combinatorially (except Y_∞) but the set has no symmetries.
Extension of the $1, 2, \ldots, q/p$, $q = p^h$, $h > 1$ special directions results to multisets: ADRIAENSEN, SzT, WEINER
Note that for multisets, the weights are modulo p (sometimes integers between 0 and $p - 1$, sometimes integers between $-(p - 1)/2$ and $(p - 1)/2$, sometimes field element).
S. Adriaensen, T. Szőnyi, Zs. Weiner, Multisets with few special directions and small weight codewords in Desarguesian planes, arXiv:2411.19201

In the eighties ERDŐS proved posed (via J. LEHEL the following question: colour the pts of Π_q with two colours. A line is balanced it it has the same no. of red/blue pts (q odd).

What is the max. no. of balanced lines?

(From discrepancy lower bound we see that there are unbalanced lines.) We want a lower bound for the no. of unbalanced lines.

Easy: there are at least $q+1$ unbalanced lines. If less, then there is a pt which not on unbalanced lines, say it is red. Then the no. of red pts is $(q^2+1)/2$. There is also a point which is on exactly 1 unbalanced line. It cannot be red, and if it is blue, then the no. of red pts. is at least $q(q+1)/2$, contradiction.

A slight refinement shows that for $q+1$ unbalanced lines they have to have a common point and the lines through this point are monochromatic (except perhaps this point).

We are still checking our manuscript, so corrections may be necessary (and in the worst case, parts of the proofs may collapse).

A natural generalization

Definition

Colour the points of $\mathrm{PG}(2,q)$ with red and blue and fix an integer $0 \leq r \leq q+1$. Let $b = q+1-r$ and assume that $r \leq b$ The line ℓ in $\mathrm{PG}(2,q)$ is (r,b)-*coloured*, if the number of red points on ℓ is r, while the number of blue points is b.

(r,b)-coloured lines are *balanced lines* and all other lines are *unbalanced lines*. Erdős' original question $r=b$.
Then previous proof can be copied for the more general case if $r>1$. Trivial colouring: r lines through a point P are entirely red, b lines entirely blue (except P).

Stability version

What is the next possible number of unbalanced lines? (assuming the colouring non-triv.)
Combinatorially (that is for Π_q):
MARKÓ: at least $\frac{13}{8}q$ (in case of the Erdős version)
at least $\frac{4}{3}q$ in the general case (if $r > 1$) with two exceptions (lines of a dual hyperoval are completely blue, or one line of the dual hyperoval is blue, the rest of the remaining lines of the hyperoval are red).
Next easy example: change the colour of one point in the triv. col. (in this case there are $2q+1$ unbalanced lines).
We try to prove this bound for $PG(2, q)$ (for general r, b), sometimes under extra conditions.
Á. Markó, Nonuniform lines in finite projective planes, *Art of Discr. and Appl. Math.*, accepted

There is a 0-point: i.e. with only balanced lines

In this case we need to assume that there is a line with $\neq r \bmod p$ red points. So, choose such a line as the line at infty, identify $\mathrm{AG}(2,q)$ with $\mathrm{GF}(q^2)$. Using the nucleus polynomial (cf. BLOKHUIS, WILBRINK)

$$F(X) = \sum_{a \in R'} (X-a)^{q-1} + a_1^{q-1} + \dots a_k^{q-1},$$

(where R' is the set of red pts in $\mathrm{AG}(2,q)$, and $a_1, \dots, a_k$ are the red pts on the line at infty. Using this, we can show that there are few (at most $q-1$) 0-points. Through the other points there are at least 2 unbalanced lines, so we are already close to $2q$ unbalanced lines. With some extra work, we can show that there have to be at least $2q+2$ unbalanced lines in this case.
Remark: In the original Erdős case, the mod p condition is either satisfied automatically, or we can show that there are more than $2q+2$ unbalanced lines.

Case 2: Unbalanced lines form a dual blocking set

Theorem

Colour the points of the plane $PG(2,q)$ either red or blue. A line is balanced if it contains r red and b blue points, where $0 < r \leq b < q+1$. Assume unbalanced lines form a dual blocking set and there are less than $2q+1$. Then, the red points form one of the following structures:

1. *a line and a point not on the line ($r = 1$),*
2. *r concurrent lines,*
3. *r concurrent lines minus their intersection,*
4. *a Baer subplane ($r = 1$).*

Very short sketch of the proof

Proof uses old results by TALLINI SCAFATI on bl. sets of type $(1, n)$. Also the fact that a bl. set of size $\leq 2q$ contains a unique minimal bl. set HARRACH, SzT.
Note that there are non-trivial colourings with $2q+1$ unbalanced lines, for example take a Baer subplane and at one of its points $r-1$ tangents and colour them red. If $r = \sqrt{q}$ or $\sqrt{q}+1$, then this example gives $2q - \sqrt{q}$ unbalanced lines, showing that the above mod p condition is necessary (if there is a 0-point).

Of course, when p is prime, then the above mod p condition is automatical.
We can also use the results of the paper
T. Szőnyi, Zs. Weiner, Stability of k mod p multisets and small weight codewords of the code generated by the lines of $\mathrm{PG}(2, q)$, *J. Combin. Theory Ser. A* **157** (2018), 321–333
to guarantee that the not r mod p lines go through some points (if there are not too many of them). It also indicates the connection with small weight codewords.

Connection with codewords

Consider a red point. It is a line of the dual plane. Add up the characteristic vectors of these lines of the dual plane (corresponding to red points). What is the coordinate corresponding to a point of the dual plane, of this sum? The point of the dual plane corresponds to a line and the coordinate is the size of intersection of this line and the red points. So, for balanced lines, it is r. Subtract $r\mathbf{j}$ (the all-1 vector). Then we get a vector in the code generated by lines of the dual plane, which is zero many times. Its weight is the number of unbalanced lines.
For example, the Kiss-Somlai example corresponds to a Bagchi type codeword (whose support is contained in the union of 3 concurrent lines). See also the Adriaensen, SzT, Weiner manuscript on arXiv, where we use the term "‘odd codeword" instead of Bagchi type codeword. This connection can also be found there.

BAGCHI: constructed a codeword of weight $3p - 3$ explicitly. Its support is contained in the union of 3 lines and it is also in the dual code.
Found independently by DE BOECK and VANDENDRIESSCHE. They also described it as a linear combination, which roughly speaking corresponds to finding a multiset in the original plane such that the above procedure gives the codeword. In general, it is not obvious when such a multiset is a set.

Consider a coloring of the points of $\mathrm{PG}(2,p)$, where $p \geq 19$, into red and blue. A line is called balanced if it contains exactly $0 < r < p+1$ red points. Suppose that there are at most $\max\{3p+1, 4p-22\}$ unbalanced lines. Then either the set of red points or the set of blue points has one of the following structures:

(1) r concurrent lines through a point P, the color of P might be switched (but not necessarily).

(2) The structure described in case (1), with switching the color of at most two points different from P.

(3) One line along with three points outside of it.

(4) A set equivalent, up to an affine transformation, to S or its complement in Example 6, union with a subset (which may be empty or the entire set) of the directions $\{(0),(1),(\infty)\}$.

(5) a triangle, where the vertices have arbitrary colours (for $r=3$).

Also, all the listed structure yields a coloring with at most $\max\{3p+1, 4p-22\}$ unbalanced lines.

Theorem

Colour the points of $\mathrm{PG}(2,p)$, *p prime, red and blue. We say that a line is balanced if it contains exactly r red points. Suppose that* $r < c\sqrt{p}$ *and the number of unbalanced lines,* δ, *is less than* $d\sqrt{p}(p+1)$, *where* $c, d > 0$ *and* $8d(c+d) + \frac{20c+28d}{\sqrt{p}} + \frac{19}{p} < 1$ *hold. Then we can change the color of at most* $\lceil d\sqrt{p} \rceil$ *points so that the coloring we obtain contains r completely red lines and all other points are blue. Furthermore, the size of the union of the set of recolored points and the intersection points of the r red lines is at most* $\lceil d\sqrt{p} \rceil$.

Thank you for your attention!